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RESEARCH ARTICLE

On some properties of generalized Fibonacci and Lucas polynomials

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ABSTRACT

In this paper we investigate some properties of generalized Fibonacci and Lucas polynomials. We give some new identities using matrices and Laplace expansion for the generalized Fibonacci and Lucas polynomials. Also, we introduce new families of tridiagonal matrices whose successive determinants generate any subsequence of these polynomials.

1. Introduction

In [16], $h(x)$-Fibonacci polynomials are defined by $F_{h,0}(x) = 0$, $F_{h,1}(x) = 1$ and $F_{h,n+1}(x) = h(x)F_{h,n}(x) + F_{h,n-1}(x)$ for $n \geq 1$. $h(x)$-Lucas polynomials are defined by $L_{h,0}(x) = 2$, $L_{h,1}(x) = h(x)$ and $L_{h,n+1}(x) = h(x)L_{h,n}(x) + L_{h,n-1}(x)$ for $n \geq 1$. Therefore some properties of these polynomials are presented in that paper.

Let $p(x)$ and $q(x)$ be polynomials with real coefficients, $p\,(x) \neq 0$, $q\,(x) \neq 0$ and $p^2\,(x) + 4q\,(x) > 0$. In [9], it was defined generalized Fibonacci polynomials $F_{p,q,n}(x)$ as

$$F_{p,q,n+1}(x) = p(x)F_{p,q,n}(x) + q(x)F_{p,q,n-1}(x), \quad n \geq 1 \tag{1}$$

with initial values $F_{p,q,0}(x) = 0, F_{p,q,1}(x) = 1$ and generalized Lucas polynomials $L_{p,q,n}(x)$ as

$$L_{p,q,n+1}(x) = p(x)L_{p,q,n}(x) + q(x)L_{p,q,n-1}(x), \quad n \geq 1 \tag{2}$$

with the initial values $L_{p,q,0}(x) = 2, L_{p,q,1}(x) = p(x)$. In that paper, it was derived factorizations and representations of polynomial analogue of an arbitrary binary sequence by matrix methods. In [11], it was given factorizations of Pascal matrix involving $(p, q)-$Fibonacci polynomials. In [19], it was obtained some arithmetic and combinatorial identities for the $(p, q)-$Fibonacci and Lucas polynomials. In Section 2, we obtain some basic properties of generalized Fibonacci and Lucas polynomials. In Section 3, we give some properties of these polynomials using 2×2 matrices. In Section 4, we make the proof of two idenitites concerning generalized Fibonacci and Lucas polynomials using Laplace expansion of determinants. In Section 5, we give new families of tridiagonal matrices whose successive determinants generate any subsequence of the generalized Fibonacci and Lucas polynomials.

2. Generalized Fibonacci and Lucas polynomials

Let $p(x)$ and $q(x)$ be polynomials with real coefficients, $p\,(x) \neq 0$, $q\,(x) \neq 0$ and $p^2\,(x) + 4q\,(x) > 0$. In this section, firstly we consider the generalized Fibonacci polynomials $F_{p,q,n}(x)$ defined in (1). The first six generalized Fibonacci polynomials are given in the following table :

$F_{p,q,1}(x) = 1$
$F_{p,q,2}(x) = p(x)$
$F_{p,q,3}(x) = p^2(x) + q(x)$
$F_{p,q,4}(x) = p^3(x) + 2p(x)q(x)$
$F_{p,q,5}(x) = p^4(x) + 3p^2(x)q(x) + q^2(x)$
$F_{p,q,6}(x) = p^5(x) + 4p^3(x)q(x) + 3p(x)q^2(x).$

For $p(x) = x$ and $q(x) = 1$ we have Catalan's Fibonacci polynomials $F_n(x)$; for $p(x) = 2x$ and $q(x) = 1$ we have Byrd's polynomials $\varphi_n(x)$; for $p(x) = k$ and $q(x) = t$ we have generalized Fibonacci numbers U_n ; for $p(x) = k$ and $q(x) = 1$ we have k-Fibonacci numbers $F_{k,n}$; for $p(x) = q(x) = 1$ we have classical Fibonacci numbers F_n (for more details see [2], [4], [8], [10], [18] and the references therein).

The generating function $g_{F,p,q}(t)$ of the generalized Fibonacci polynomials $F_{p,q,n}(x)$ is defined by

$$g_{F,p,q}(t) = \sum_{n=0}^{\infty} F_{p,q,n}(x)t^n. \tag{3}$$

From [11], we know that the generating function of the generalized Fibonacci polynomials $F_{p,q,n}(x)$ is

$$g_{F,p,q}(t) = \frac{t}{1 - tp(x) - t^2q(x)}. \tag{4}$$

Theorem 1. *Assume that $p(x)$ is an odd polynomial and $q(x)$ is an even polynomial. Then $F_{p,q,n}(-x) = (-1)^{n+1}F_{p,q,n}(x)$ for $n \geq 0$.*

Proof. From (3), and (4), we have

$$\sum_{n=0}^{\infty} F_{p,q,n}(-x)(-t)^n = \frac{-t}{1 - tp(x) - t^2q(x)}$$

and

$$\begin{aligned}\sum_{n=0}^{\infty}(-1)^{n+1}F_{p,q,n}(-x)t^n &= \frac{t}{1 - tp(x) - t^2q(x)} \\ &= \sum_{n=0}^{\infty} F_{p,q,n}(x)t^n.\end{aligned}$$

Then the proof is follows.

□

Binet's formulas are well known among the Fibonacci numbers. Let $\alpha(x)$ and $\beta(x)$ be the roots of the characteristic equation

$$v^2 - vp(x) - q(x) = 0, \tag{5}$$

of the recurrence relation (1). From [9], we know that

$$F_{p,q,n}(x) = \frac{\alpha^n(x) - \beta^n(x)}{\alpha(x) - \beta(x)}, \text{ for } n \geq 0, \tag{6}$$

where

$$\left.\begin{aligned}\alpha(x) &= \tfrac{p(x)+\sqrt{p^2(x)+4q(x)}}{2}, \\ \beta(x) &= \tfrac{p(x)-\sqrt{p^2(x)+4q(x)}}{2}.\end{aligned}\right\} \tag{7}$$

Notice that $\alpha(x)+\beta(x) = p(x)$, $\alpha(x)\beta(x) = -q(x)$ and $\alpha(x) - \beta(x) = \sqrt{p^2(x) + 4q(x)}$.

Theorem 2. *For $n \geq 1$, we have*

$$F_{p,q,n}(x) = 2^{1-n}\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2j+1}p^{n-2j-1}(x)(p^2(x) + 4q(x))^j.$$

Proof. From (7), we have

$$\begin{aligned}&\alpha^n(x) - \beta^n(x) = 2^{-n}[(p(x) + \sqrt{p^2(x) + 4q(x)})^n \\ &-(p(x) - \sqrt{p^2(x) + 4q(x)})^n] \\ &= 2^{-n}[\sum_{j=0}^{n}\binom{n}{j}p^{n-j}(x)(\sqrt{p^2(x) + 4q(x)})^j \\ &-\sum_{j=0}^{n}\binom{n}{j}p^{n-j}(x)(-\sqrt{p^2(x) + 4q(x)})^j] \\ &= 2^{-n+1}\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2j+1}p^{n-2j-1}(x)(\sqrt{p^2(x) + 4q(x)})^{2j+1}.\end{aligned}$$

From the equation (6), then we obtain

$$\begin{aligned}F_{p,q,n}(x) &= \tfrac{\alpha^n(x)-\beta^n(x)}{\alpha(x)-\beta(x)} = \tfrac{\alpha^n(x)-\beta^n(x)}{\sqrt{p^2(x)+4q(x)}} \\ &= 2^{-n+1}\sum_{j=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n}{2j+1}p^{n-2j-1}(x)(p^2(x) + 4q(x))^j.\end{aligned}$$

□

In [12], definitions of Chebyshev polynomials of the first and second kinds are given by the followings (resp.)

$$T_n(x) = \cos n\theta \text{ and } H_n(x) = \frac{\sin[(n+1)\theta]}{\sin\theta},$$

where $x = \cos\theta$, $0 \leq \theta \leq \pi$.

We know that the generating functions of Chebyshev polynomials of the first and second kinds are

$$\sum_{n=0}^{\infty} T_n(t)\, z^n = \frac{1 - tz}{1 - 2tz + z^2}$$

and

$$\sum_{n=0}^{\infty} H_n(t) z^n = \frac{1}{1-2tz+z^2},$$

respectively. Also we can write Chebyshev polynomials of the first and second kinds as follows:

$$T_n(t) = \frac{n}{2}\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^j}{n-j} \binom{n-j}{j} (2t)^{n-2j}$$

with $T_0(t) = 1$ and

$$H_n(t) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} (2t)^{n-2j}$$

with $H_0(t) = 1$ (for more details one can see [3], [13] and [17]).

Theorem 3. *For $n \geq 1$, we have*

$$F_{p,q,n}(x) = i^{n-1} q(x)^{\frac{n-1}{2}} H_{n-1}\left(\frac{p(x)}{2i\sqrt{q(x)}}\right),$$

where $i^2 = -1$ and

$$H_n(t) = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \binom{n-j}{j} (2t)^{n-2j}$$

with $H_0(t) = 1$ is the Chebyshev polynomial of the second kind.

Proof. We know that the generating function for the second kind Chebyshev polynomial $H_n(t)$ is

$$\sum_{n=0}^{\infty} H_n(t) z^n = \frac{1}{1-2tz+z^2}.$$

Let $z = iy\sqrt{q(x)}$ and $t = \frac{p(x)}{2i\sqrt{q(x)}}$. Then we get

$$\sum_{n=0}^{\infty} i^n y^n q(x)^{\frac{n}{2}} H_n\left(\frac{p(x)}{2i\sqrt{q(x)}}\right) = \frac{1}{1-yp(x)-y^2q(x)}$$

or

$$\sum_{n=0}^{\infty} i^n y^{n+1} q(x)^{\frac{n}{2}} H_n\left(\frac{p(x)}{2i\sqrt{q(x)}}\right) = \frac{y}{1-yp(x)-y^2q(x)}.$$

From the equation (4), we find

$$F_{p,q,n}(x) = i^{n-1} q(x)^{\frac{n-1}{2}} H_{n-1}\left(\frac{p(x)}{2i\sqrt{q(x)}}\right).$$

□

Now, we consider the generalized Lucas polynomials $L_{p,q,n}(x)$ defined in (2). The first six generalized Lucas polynomials are given in the following table :

$$\begin{aligned}
L_{p,q,1}(x) &= p(x)\\
L_{p,q,2}(x) &= p^2(x) + 2q(x)\\
L_{p,q,3}(x) &= p^3(x) + 3p(x)q(x)\\
L_{p,q,4}(x) &= p^4(x) + 4p^2(x)q(x) + 2q^2(x)\\
L_{p,q,5}(x) &= p^5(x) + 5p^3(x)q(x) + 5p(x)q^2(x)\\
L_{p,q,6}(x) &= p^6(x) + 6p^4(x)q(x)\\
&\quad + 9p^2(x)q^2(x) + 2q^3(x).
\end{aligned}$$

For $p(x) = x$ and $q(x) = 1$ we have Lucas polynomials $L_n(x)$; for $p(x) = k$ and $q(x) = t$ we have generalized Lucas numbers V_n; for $p(x) = k$ and $q(x) = 1$ we have k-Lucas numbers $L_{k,n}$; for $p(x) = q(x) = 1$ we have classical Lucas numbers L_n (for more details see [5], [7], [10], [18] and the references therein).

The generating function $g_{L,p,q}(t)$ of the Lucas polynomials $L_{p,q,n}(x)$ is defined by

$$g_{L,p,q}(t) = \sum_{n=0}^{\infty} L_{p,q,n}(x) t^n.$$

From [11], we know that the generating function of the generalized Lucas polynomials $L_{p,q,n}(x)$ is

$$g_{L,p,q}(t) = \frac{2-tp(x)}{1-tp(x)-t^2q(x)}. \tag{8}$$

Theorem 4. *Assume that $p(x)$ is an odd polynomial and $q(x)$ is an even polynomial. Then we have*

$$L_{p,q,n}(-x) = (-1)^n L_{p,q,n}(x), \text{ for } n \geq 0.$$

Proof. Using the equation (8), the proof is clear. □

From [9], we know that Binet's formula for $L_{p,q,n}(x)$ is

$$L_{p,q,n}(x) = \alpha^n(x) + \beta^n(x) \text{ for } n \geq 0,$$

where $\alpha(x)$ and $\beta(x)$ are the roots of the characteristic equation (5). Using Binet formulas for the generalized Fibonacci and Lucas polynomials, we obtain the following corollaries.

Corollary 1. *For $n \geq 0$, we have*

$$L_{p,q,n}(x) = p(x) F_{p,q,n}(x) + 2q(x) F_{p,q,n-1}(x). \tag{9}$$

Corollary 2. *For $n \geq 0$, we have*

$$\alpha^n(x) = \frac{L_{p,q,n}(x) + \sqrt{p^2(x)+4q(x)}F_{p,q,n}(x)}{2}$$

and

$$\beta^n(x) = \frac{L_{p,q,n}(x) - \sqrt{p^2(x)+4q(x)}F_{p,q,n}(x)}{2}.$$

Corollary 3. *For $n \geq 0$, we have*

$$L^2_{p,q,n}(x) - (p^2(x)+4q(x))F^2_{p,q,n}(x) = 4q(x)(-1)^n.$$

Corollary 4. *For $n \geq 0$, we have*

$$F_{p,q,2n}(x) = F_{p,q,n}(x)L_{p,q,n}(x).$$

As similar to Theorem 3, we can give the following theorem giving the relation between $L_{p,q,n}(x)$ and $T_n(x)$. Since its proof is similar to that of Theorem 3, we omit it.

Theorem 5. *For $n \geq 0$, we have*

$$L_{p,q,n}(x) = 2i^n q(x)^{\frac{n}{2}} T_n\left(\frac{p(x)}{2i\sqrt{q(x)}}\right),$$

where $i^2 = -1$ and

$$T_n(t) = \frac{n}{2}\sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \frac{(-1)^j}{n-j}\begin{pmatrix} n-j \\ j \end{pmatrix}(2t)^{n-2j}$$

with $T_0(t) = 1$ is the Chebyshev polynomial of the first kind.

3. Some new identities for generalized Fibonacci and Lucas polynomials

In [19], it was defined generalized Fibonacci and Lucas polynomials with negative subscript of the following form:

$$\left.\begin{array}{r} F_{p,q,-n}(x) = \frac{-F_{p,q,n}(x)}{(-q(x))^n}, \\ L_{p,q,-n}(x) = \frac{L_{p,q,n}(x)}{(-q(x))^n}. \end{array}\right\} \qquad (10)$$

In this section we find some identities using the following 2×2 matrices

$$A = \begin{bmatrix} p(x) & q(x) \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ q(x) & p(x) \end{bmatrix}.$$

Indeed the above matrices satisfy $X^2 = p(x)X + q(x)I$. We obtain some new identities using 2×2 matrices of the form

$$X^2 = p(x)X + q(x)I. \qquad (11)$$

In the following theorems we use the proof methods like as [18].

Theorem 6. *If X is a square matrix of the form $X^2 = p(x)X + q(x)I$, then we have*

$$X^n = F_{p,q,n}(x)X + q(x)F_{p,q,n-1}(x)I,$$

for any integer n.

Proof. It can be easily seen that $X^n = F_{p,q,n}(x)X + q(x)F_{p,q,n-1}(x)I$ for every $n \in N$ using mathematical induction. Now we show that $X^{-n} = F_{p,q,-n}(x)X + q(x)F_{p,q,-n-1}(x)I$ for every $n \in N$. Let $K = p(x)I - X$, then we have

$$\begin{aligned} K^2 &= (p(x)I - X)^2 \\ &= p^2(x)I - p(x)X + q(x)I \\ &= p(x)K + q(x)I. \end{aligned}$$

So we get $K^n = F_{p,q,n}(x)K + q(x)F_{p,q,n-1}(x)I$. Then

$$\begin{aligned} &(-q(x))^n X^{-n} = K^n \\ &= F_{p,q,n}(x)K + q(x)F_{p,q,n-1}(x)I \\ &= F_{p,q,n}(x)\left(p(x)I - X\right) \\ &+q(x)F_{p,q,n-1}(x)I \\ &= F_{p,q,n+1}(x)I - F_{p,q,n}(x)X. \end{aligned}$$

Thus using the equation (10), we find

$$X^{-n} = \frac{-F_{p,q,n}(x)X}{(-q(x))^n} + \frac{F_{p,q,n+1}(x)I}{(-q(x))^n}$$

and

$$X^{-n} = F_{p,q,-n}(x)X + q(x)F_{p,q,-n-1}(x)I.$$

□

Theorem 7. *Let X be an arbitrary 2×2 matrix. Then $X^2 = p(x)X + q(x)I$ if and only if X is of the form*

$$X = \begin{bmatrix} a & b \\ c & p(x) - a \end{bmatrix}, \text{ with } detX = -q(x)$$

or $X = \delta I$ where $\delta \in \{\alpha(x), \beta(x)\}$, $\alpha(x) = \frac{p(x)+\sqrt{p^2(x)+4q(x)}}{2}$ and $\beta(x) = \frac{p(x)-\sqrt{p^2(x)+4q(x)}}{2}$.

Proof. Assume that $X^2 = p(x)X + q(x)I$. Then the minimal polynomial of X must divide $\lambda^2 - \lambda p(x) - q(x)$. So it must be $\lambda - \alpha(x)$, $\lambda - \beta(x)$ or $\lambda^2 - \lambda p(x) - q(x)$. In the first case $X = \alpha(x)I$, in the second case $X = \beta(x)I$ and in the third case characteristic polynomial of X should be $\lambda^2 - \lambda p(x) - q(x)$ because X is a 2×2 matrix. Clearly, its trace is $p(x)$ and its determinant is $-q(x)$. The rest of the proof can be similarly completed. □

Corollary 5. *If* $X =$
$$\begin{bmatrix} a & b \\ c & p(x) - a \end{bmatrix}$$
is a matrix with $detX = -q(x)$, *then we have*
$$X^n = \begin{bmatrix} aF_{p,q,n}(x) + q(x)F_{p,q,n-1}(x) & bF_{p,q,n}(x) \\ cF_{p,q,n}(x) & F_{p,q,n+1}(x) - aF_{p,q,n}(x) \end{bmatrix}.$$

Proof. From Theorem 7, we know that $X^2 = p(x)X + q(x)I$. Then, from Theorem 6 we get $X^n = F_{p,q,n}(x)X + q(x)F_{p,q,n-1}(x)I$ for any integer n. Then the proof follows. □

Corollary 6. *Let* $S = \begin{bmatrix} \frac{p(x)}{2} & \frac{p^2(x)+4q(x)}{2} \\ \frac{1}{2} & \frac{p(x)}{2} \end{bmatrix}$, *then we have*
$$S^n = \begin{bmatrix} \frac{L_{p,q,n}(x)}{2} & \frac{\left(p^2(x)+4q(x)\right)F_{p,q,n}(x)}{2} \\ \frac{F_{p,q,n}(x)}{2} & \frac{L_{p,q,n}(x)}{2} \end{bmatrix}.$$

Proof. Since $S^2 = p(x)S + q(x)I$, the proof is completed by using Corollary 5. □

4. Generalized Fibonacci and Lucas polynomials with Laplace expansion

In [6], it was given some identities about Fibonacci numbers using Laplace expansion. In this section we give two theorems about generalized Fibonacci and Lucas polynomials and prove them using Laplace expansion of determinants.

Let us consider the $n \times n$ tridiagonal matrix $C(n)$ defined by the following form:

$$C(n) = \begin{pmatrix} p(x) & i\sqrt{q(x)} & & & \\ i\sqrt{q(x)} & p(x) & i\sqrt{q(x)} & & \\ & i\sqrt{q(x)} & p(x) & . & \\ & & . & . & . \\ & & & . & . & i\sqrt{q(x)} \\ & & & & i\sqrt{q(x)} & p(x) \end{pmatrix}.$$

Theorem 8. *For any integer* k $(2 \le k \le n-1)$, *we have*

$$F_{p,q,n}(x) = F_{p,q,k}(x)F_{p,q,n-k+1}(x) + q(x)F_{p,q,k-1}(x)F_{p,q,n-k}(x). \qquad (12)$$

Proof. From $k = 2$ to $k = n-1$, the equation (12) becomes the followings:

$$\begin{aligned} F_{p,q,n}(x) &= F_{p,q,2}(x)F_{p,q,n-1}(x) \\ &\quad + q(x)F_{p,q,1}(x)F_{p,q,n-2}(x), \\ F_{p,q,n}(x) &= F_{p,q,3}(x)F_{p,q,n-2}(x) \\ &\quad + q(x)F_{p,q,2}(x)F_{p,q,n-3}(x), \\ &\cdots \\ F_{p,q,n}(x) &= F_{p,q,n-2}(x)F_{p,q,3}(x) \\ &\quad + q(x)F_{p,q,n-3}(x)F_{p,q,2}(x), \\ F_{p,q,n}(x) &= F_{p,q,n-1}(x)F_{p,q,2}(x) \\ &\quad + q(x)F_{p,q,n-2}(x)F_{p,q,1}(x). \end{aligned}$$

It can be easily seen that $F_{p,q,n}(x) = |C(n-1)|$ for $n \ge 2$. Using Lemma 1 in [1] we get

$$\begin{aligned} &|C(n-1)| \\ &= p(x)\,|C(n-2)| + q(x)\,|C(n-3)| \\ &= p(x)F_{p,q,n-1}(x) + q(x)F_{p,q,n-2}(x) \\ &= F_{p,q,2}(x)F_{p,q,n-1}(x) + q(x)F_{p,q,n-2}(x) \end{aligned}$$

Then we find

$$F_{p,q,n}(x) = F_{p,q,2}(x)F_{p,q,n-1}(x) + q(x)F_{p,q,n-2}(x)$$

Now we use the techniques in [14] to find the determinant of the matrix $C(n-1)$. If we choose the first two rows of $C(n-1)$, there are only three 2×2 submatrices of $C(n-1)$ whoose determinants are not equal to zero.

$$\begin{aligned} &C([1,2],[1,2]) = \begin{vmatrix} p(x) & i\sqrt{q(x)} \\ i\sqrt{q(x)} & p(x) \end{vmatrix} \\ &= |C(2)| = F_{p,q,3}(x), \\ &C([1,2],[1,3]) = \begin{vmatrix} p(x) & 0 \\ i\sqrt{q(x)} & i\sqrt{q(x)} \end{vmatrix} \\ &= ip(x)\sqrt{q(x)}, \\ &C([1,2],[2,3]) = \begin{vmatrix} i\sqrt{q(x)} & 0 \\ p(x) & i\sqrt{q(x)} \end{vmatrix} \\ &= -q(x). \end{aligned}$$

Their corresponding cofactors are

$$\begin{aligned} &\widetilde{C}([1,2],[1,2]) = (-1)^{1+2+1+2}\,|C(n-3)| \\ &= F_{p,q,n-2}(x), \\ &\widetilde{C}([1,2],[1,3]) = (-1)^{1+2+1+3} i\sqrt{q(x)}\,|C(n-4)| \\ &= -i\sqrt{q(x)}F_{p,q,n-3}(x), \\ &\widetilde{C}([1,2],[2,3]) = 0. \end{aligned}$$

By the Laplace expansion in [14], we have

$$\begin{aligned}F_{p,q,n}(x) &= |C(n-1)| \\ &= C([1,2],[1,2])\widetilde{C}([1,2],[1,2]) \\ &+C([1,2],[1,3])\widetilde{C}([1,2],[1,3]) \\ &+C([1,2],[2,3])\widetilde{C}([1,2],[2,3]) \\ &= |C(2)|\, F_{p,q,n-2}(x) + p(x)i\sqrt{q(x)} \\ &(-i\sqrt{q(x)})F_{p,q,n-3}(x) + (-q(x)).0 \\ &= F_{p,q,3}(x)F_{p,q,n-2}(x) \\ &+p(x)q(x)F_{p,q,n-3}(x).\end{aligned}$$

Then we get

$$\begin{aligned}F_{p,q,n}(x) = F_{p,q,3}(x)F_{p,q,n-2}(x) \\ + q(x)F_{p,q,2}(x)F_{p,q,n-3}(x).\end{aligned}$$

If we choose the first three rows of $C(n-1)$, there are only four 3×3 submatrices of $C(n-1)$ whoose determinants are not equal to zero.

$$C([1,2,3],[1,2,3]) = \begin{vmatrix} p(x) & i\sqrt{q(x)} & 0 \\ i\sqrt{q(x)} & p(x) & i\sqrt{q(x)} \\ 0 & i\sqrt{q(x)} & p(x)\end{vmatrix}$$
$$= |C(3)| = F_{p,q,4}(x),$$
$$C([1,2,3],[1,2,4]) = \begin{vmatrix} p(x) & i\sqrt{q(x)} & 0 \\ i\sqrt{q(x)} & p(x) & 0 \\ 0 & i\sqrt{q(x)} & i\sqrt{q(x)}\end{vmatrix}$$
$$= i\sqrt{q(x)}\,|C(2)| = i\sqrt{q(x)}F_{p,q,3}(x),$$
$$C([1,2,3],[1,3,4]) = \begin{vmatrix} p(x) & 0 & 0 \\ i\sqrt{q(x)} & i\sqrt{q(x)} & 0 \\ 0 & p(x) & i\sqrt{q(x)}\end{vmatrix}$$
$$= -p(x)q(x),$$
$$C([1,2,3],[2,3,4]) = \begin{vmatrix} i\sqrt{q(x)} & 0 & 0 \\ p(x) & i\sqrt{q(x)} & 0 \\ i\sqrt{q(x)} & p(x) & i\sqrt{q(x)}\end{vmatrix}$$
$$= -i\sqrt{q(x)}q(x).$$

Their corresponding cofactors are

$$\begin{aligned}&\widetilde{C}([1,2,3],[1,2,3]) = (-1)^{6+6}\,|C(n-4)| \\ &= F_{p,q,n-3}(x), \\ &\widetilde{C}([1,2,3],[1,2,4]) = (-1)^{6+7}i\sqrt{q(x)}\,|C(n-5)| \\ &= -i\sqrt{q(x)}F_{p,q,n-4}(x), \\ &\widetilde{C}([1,2,3],[1,3,4]) = 0, \\ &\widetilde{C}([1,2,3],[2,3,4]) = 0.\end{aligned}$$

By the Laplace expansion in [14], we have

$$\begin{aligned}F_{p,q,n}(x) &= |C(n-1)| \\ &= C([1,2,3],[1,2,3])\widetilde{C}([1,2,3],[1,2,3]) \\ &+C([1,2,3],[1,2,4])\widetilde{C}([1,2,3],[1,2,4]) \\ &+C([1,2,3],[1,3,4])\widetilde{C}([1,2,3],[1,3,4]) \\ &+C([1,2,3],[2,3,4])\widetilde{C}([1,2,3],[2,3,4]) \\ &= F_{p,q,4}(x)F_{p,q,n-3}(x) \\ &+i\sqrt{q(x)}F_{p,q,3}(x)(-i)\sqrt{q(x)}F_{p,q,n-4}(x).\end{aligned}$$

Then we get

$$\begin{aligned}F_{p,q,n}(x) = F_{p,q,4}(x)F_{p,q,n-3}(x) \\ + q(x)F_{p,q,3}(x)F_{p,q,n-4}(x).\end{aligned}$$

By the mathematical induction, we prove the other identities in the equation (12). □

Let $D(n)$ be the $n\times n$ tridioganal matrix given of the following form:

$$D(n) = \begin{pmatrix} \frac{p(x)}{2} & i\sqrt{q(x)} & & & & \\ i\sqrt{q(x)} & p(x) & i\sqrt{q(x)} & & & \\ & i\sqrt{q(x)} & p(x) & . & & \\ & & . & . & . & \\ & & & . & . & i\sqrt{q(x)} \\ & & & & i\sqrt{q(x)} & p(x)\end{pmatrix}$$

Theorem 9. *For any integer k $(1\le k\le n-1)$, we have*

$$\begin{aligned}L_{p,q,n}(x) = L_{p,q,k}(x)F_{p,q,n-k+1}(x) \\ + q(x)L_{p,q,k-1}(x)F_{p,q,n-k}(x).\end{aligned} \qquad (13)$$

Proof. From $k=1$ to $k=n-1$, the equation (13) becomes the followings:

$$\begin{aligned}L_{p,q,n}(x) &= L_{p,q,1}(x)F_{p,q,n}(x) \\ &\quad + q(x)L_{p,q,0}(x)F_{p,q,n-1}(x), \\ L_{p,q,n}(x) &= L_{p,q,2}(x)F_{p,q,n-1}(x) \\ &\quad + q(x)L_{p,q,1}(x)F_{p,q,n-2}(x), \\ &\cdots \\ L_{p,q,n}(x) &= L_{p,q,n-2}(x)F_{p,q,3}(x) \\ &\quad + q(x)L_{p,q,n-3}(x)F_{p,q,2}(x), \\ L_{p,q,n}(x) &= L_{p,q,n-1}(x)F_{p,q,2}(x) \\ &\quad + q(x)L_{p,q,n-2}(x)F_{p,q,1}(x).\end{aligned}$$

It is clear that $L_{p,q,n}(x) = 2\,|D(n)|$, for $n\ge 1$. From the Corollary 1, we have $L_{p,q,n}(x) = p(x)F_{p,q,n}(x) + 2q(x)F_{p,q,n-1}(x)$. Then we get

$$\begin{aligned}L_{p,q,n}(x) = L_{p,q,1}(x)F_{p,q,n}(x) \\ + q(x)L_{p,q,0}(x)F_{p,q,n-1}(x).\end{aligned}$$

The rest of the proof can be completed similar to the proof of the Theorem 8. □

In [19], for $m = 0$ in the equation (3.9) coincides with our Theorem 9 for $k = n - 1$.

5. Generalized Fibonacci and generalized Lucas polynomials subsequences

In this section we obtain another applications of Lemma 1 in [1]. We generalize the family of tridiagonal matrices to a subsequence of generalized Fibonacci (resp. generalized Lucas) polynomials which is a family of tridiagonal matrices whose successive determinants are given by that polynomials. To do this, we use the following identities. For $n \geq 1$ we have

$$F_{p,q,m+n}(x) = L_{p,q,n}(x)F_{p,q,m}(x) + (-1)^{n+1}q^n(x)F_{p,q,m-n}(x) \tag{14}$$

and

$$L_{p,q,m+n}(x) = L_{p,q,n}(x)L_{p,q,m}(x) + (-1)^{n+1}q^n(x)L_{p,q,m-n}(x). \tag{15}$$

These identities was proved in [15] for $p(x) = k$ and $q(x) = 1$. We give the following theorems using the proof methods given in [1].

Theorem 10. *Let $M_{\alpha,\beta}(n), n = 1, 2, ...$ be the family of symmetric tridiagonal matrices whose elements satisfy following conditions :*

$$\begin{array}{l} m_{1,1} = F_{p,q,\alpha+\beta}(x), \\ m_{2,2} = \left\lceil \frac{F_{p,q,2\alpha+\beta}(x)}{F_{p,q,\alpha+\beta}(x)} \right\rceil, \\ m_{1,2} = m_{2,1} \\ = \sqrt{m_{2,2}F_{p,q,\alpha+\beta}(x) - F_{p,q,2\alpha+\beta}(x)}, \\ m_{j,j+1} = m_{j+1,j} = \\ \sqrt{(-1)^\alpha q^\alpha(x)},\ 2 \leq j \leq 3, \\ m_{j,j} = L_{p,q,\alpha}(x),\ 3 \leq j \leq k, \end{array}$$

with $\alpha \in \mathbb{Z}^+$ and $\beta \in \mathbb{N}$. The successive determinants of this family of matrices is

$$|M_{\alpha,\beta}(n)| = F_{p,q,\alpha n+\beta}(x).$$

Proof. We use the principle of mathematical induction. We have

$$|M_{\alpha,\beta}(1)| = \det F_{p,q,\alpha+\beta}(x) = F_{p,q,\alpha+\beta}(x)$$

and

$$\begin{array}{l} |M_{\alpha,\beta}(2)| \\ = \begin{vmatrix} F_{p,q,\alpha+\beta}(x) & \sqrt{m_{2,2F_{p,q,\alpha+\beta}(x)-F_{p,q,2\alpha+\beta}(x)}} \\ \sqrt{m_{2,2F_{p,q,\alpha+\beta}(x)-F_{p,q,2\alpha+\beta}(x)}} & \left\lceil \frac{F_{p,q,2\alpha+\beta}(x)}{F_{,p,q,\alpha+\beta}(x)} \right\rceil \end{vmatrix} \\ = F_{p,q,2\alpha+\beta}(x). \end{array}$$

Now we assume that $|M_{\alpha,\beta}(n)| = F_{p,q,\alpha n+\beta}(x)$ for $1 \leq k \leq n$. Then by Lemma 1 in [1] and (14) we have

$$\begin{array}{l} M_{\alpha,\beta}(n+1) \\ = m_{n,n}\,|M_{\alpha,\beta}(n)| - m_{n,n-1}m_{n-1,n}\,|M_{\alpha,\beta}(n-1)| \\ = L_{p,q,\alpha}(x)\,|M_{\alpha,\beta}(n)| - (-1)^\alpha q^\alpha(x)\,|M_{\alpha,\beta}(n-1)| \\ = L_{p,q,\alpha}(x)F_{p,q,\alpha n+\beta}(x) + (-1)^{\alpha+1}q^\alpha(x)F_{p,q,\alpha n+\beta-\alpha}(x). \end{array}$$

Using the equation (14), we get

$$\begin{aligned} M_{\alpha,\beta}(n+1) &= F_{p,q,\alpha+\alpha n+\beta}(x) \\ &= F_{p,q,\alpha(n+1)+\beta}(x). \end{aligned}$$

□

Theorem 11. *Let $R_{\alpha,\beta}(n), n = 1, 2, ...$ be the family of symmetric tridiagonal matrices whose elements satisfy the following conditions :*

$$\begin{array}{l} r_{1,1} = L_{p,q,\alpha+\beta}(x), \\ r_{2,2} = \left\lceil \frac{L_{p,q,2\alpha+\beta}(x)}{L_{p,q,\alpha+\beta}(x)} \right\rceil, \\ r_{1,2} = r_{2,1} \\ = \sqrt{r_{2,2}L_{p,q,\alpha+\beta}(x) - L_{p,q,2\alpha+\beta}(x)}, \\ r_{j,j+1} = r_{j+1,j} \\ = \sqrt{(-1)^\alpha q^\alpha(x)},\ 2 \leq j \leq 3, \\ r_{j,j} = L_{p,q,\alpha}(x),\ 3 \leq j \leq k, \end{array}$$

with $\alpha \in \mathbb{Z}^+$ and $\beta \in \mathbb{N}$. The successive determinants of this family of matrices is

$$|R_{\alpha,\beta}(n)| = L_{p,q,\alpha n+\beta}(x).$$

Proof. We use the principle of mathematical induction. We have

$$|R_{\alpha,\beta}(1)| = \det L_{p,q,\alpha+\beta}(x) = L_{p,q,\alpha+\beta}(x)$$

and

$$\begin{array}{l} |R_{\alpha,\beta}(2)| \\ = \begin{vmatrix} L_{p,q,\alpha+\beta}(x) & \sqrt{m_{2,2L_{p,q,\alpha+\beta}(x)-L_{p,q,2\alpha+\beta}(x)}} \\ \sqrt{m_{2,2L_{p,q,\alpha+\beta}(x)-L_{p,q,2\alpha+\beta}(x)}} & \left\lceil \frac{L_{p,q,2\alpha+\beta}(x)}{L_{,p,q,\alpha+\beta}(x)} \right\rceil \end{vmatrix} \\ = L_{p,q,2\alpha+\beta}(x). \end{array}$$

Now we assume that $|R_{\alpha,\beta}(n)| = L_{p,q,\alpha n+\beta}(x)$ for $1 \leq k \leq n$. Then by Lemma 1 in [1] and (15) we find

$$\begin{aligned} & R_{\alpha,\beta}(n+1) \\ & = r_{n,n}\left|R_{\alpha,\beta}(n)\right| - r_{n,n-1}r_{n-1,n}\left|R_{\alpha,\beta}(n-1)\right| \\ & = L_{p,q,\alpha}(x)\left|R_{\alpha,\beta}(n)\right| - (-1)^{\alpha}q^{\alpha}(x)\left|R_{\alpha,\beta}(n-1)\right| \\ & = L_{p,q,\alpha}(x)L_{p,q,\alpha n+\beta}(x) \\ & +(-1)^{\alpha+1}q^{\alpha}(x)L_{p,q,\alpha(n-1)+\beta}(x). \end{aligned}$$

Using the equation (15), we get

$$\begin{aligned} R_{\alpha,\beta}(n+1) &= L_{p,q,\alpha+\alpha n+\beta}(x) \\ &= L_{p,q,\alpha(n+1)+\beta}(x). \end{aligned}$$

□

As a consequence of Theorem 10 and Theorem 11, we establish new families of tridiagonal matrices whose successive determinants generate any subsequence of the generalized Fibonacci and generalized Lucas polynomials. For example, we have

$$F_{p,q,4n-2}(x) = \begin{pmatrix} p(x) & 0 & 0 & & & \\ 0 & p^4(x)+4p^2(x)q(x)+3q^2(x) & q^2(x) & & & \\ 0 & q^2(x) & p^4(x)+4p^2(x)q(x)+2q^2(x) & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & q^2(x) \\ & & & & q^2(x) & p^4(x)+4p^2(x)q(x)+2q^2(x) \end{pmatrix}$$

and

$$L_{p,q,4n-2}(x) = \begin{pmatrix} p^2(x)+2q(x) & 0 & 0 & & & \\ 0 & p^4(x)+4p^2(x)q(x)+q^2(x) & q^2(x) & & & \\ 0 & q^2(x) & p^4(x)+4p^2(x)q(x)+2q^2(x) & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & \cdot & \cdot & q^2(x) \\ & & & & q^2(x) & p^4(x)+4p^2(x)q(x)+2q^2(x) \end{pmatrix}.$$

6. Conclusion

In this study we give some new properties of generalized Fibonacci and Lucas polynomials using matrices, complex numbers and Chebyshev polynomials. Our results generalize some known results in the literature.

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