On the Hermite-Hadamard-Fejer-type inequalities for co-ordinated convex functions via fractional integrals

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ABSTRACT

In this paper, using Riemann-Liouville integral operators, we establish new fractional integral inequalities of Hermite-Hadamard-Fejer type for co-ordinated convex functions on a rectangle of $\mathbb{R}^2$. The results presented here would provide extensions of those given in earlier works.

Theorem 1. Let $\Phi : [a, b] \to \mathbb{R}$ be a convex function. Then the inequality hold:

$$
\Phi \left( \frac{a + b}{2} \right) \geq \frac{1}{b - a} \int_{a}^{b} \Phi (x) \, dx \geq \frac{\Phi (a) + \Phi (b)}{2},
$$

where $\Psi : [a, b] \to \mathbb{R}$ is nonnegative, integrable and symmetric to $\frac{a + b}{2}$.

In the following, we will give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used further in this paper. More details, one can consult \cite{12,18}.

Definition 1. (\cite{4,12,18}) Let $\Phi \in L_1([a, b])$. The Riemann-Liouville integrals $J^\alpha_{a+} \Phi$ and $J^\alpha_{b-} \Phi$ of order $\alpha > 0$ with $a \geq 0$ are defined by

\begin{align*}
J^\alpha_{a+} \Phi (x) &= \frac{1}{\Gamma (\alpha)} \int_{a}^{x} (x - t)^{\alpha - 1} \Phi (t) \, dt, \\
J^\alpha_{b-} \Phi (x) &= \frac{1}{\Gamma (\alpha)} \int_{x}^{b} (t - x)^{\alpha - 1} \Phi (t) \, dt,
\end{align*}

where $\Gamma (\alpha)$ is the Gamma function.

1. Introduction

Let $\Phi : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval $I$ of real numbers and $a, b \in I$, with $a < b$. The following double inequality is well known in the literature as the Hermite-Hadamard inequality \cite{13}:

$$
\Phi \left( \frac{a + b}{2} \right) \leq \frac{1}{b - a} \int_{a}^{b} \Phi (x) \, dx \leq \frac{\Phi (a) + \Phi (b)}{2}.
$$

The most well-known inequalities related to the integral mean of a convex function are the Hermite Hadamard inequalities or its weighted versions, the so-called Hermite-Hadamard-Fejér inequalities (see, \cite{13,19,21}). In \cite{11}, Fejér gave a weighted generalization of the inequalities \cite{11} as the following:

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function with 0 is a convex function on \([a, b]\) with respect to \(L\):
\[
J\Phi(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x-t)^{-\alpha} \Phi(t)dt, \quad x > a
\]
and
\[
J^\alpha_{b-}\Phi(x) = \frac{1}{\Gamma(\alpha)} \int_{b}^{x} (t-x)^{-\alpha} \Phi(t)dt, \quad x < b
\]
respectively. Here, \(\Gamma(\alpha)\) is the Gamma function and \(J^0_{a+}\Phi(x) = J^0_{b-}\Phi(x) = \Phi(x)\).

Meanwhile, in [24], Sarikaya et al. gave the following interesting Riemann-Liouville integral inequalities of Hermite-Hadamard-type:

**Theorem 2.** Let \(K : [a, b] \to \mathbb{R}\) be a positive function with \(0 \leq a < b\) and \(K \in L_1([a, b])\). If \(K\) is a convex function on \([a, b]\), then the following inequalities for fractional integrals hold:

\[
K \left( \frac{a+b}{2} \right) \leq \frac{\Gamma(\alpha+1)}{2(b-a)^\alpha} \left[ J^\alpha_{a+}K(b) + J^\alpha_{b-}K(a) \right]
\]

with \(\alpha > 0\).

Later, in [14], Iscan presented the following Hermite-Hadamard-Fejer type inequalities for convex functions via Riemann-Liouville fractional integrals:

**Theorem 3.** Let \(K : [a, b] \to \mathbb{R}\) be convex function with \(0 \leq a < b\) and \(K \in (L_1[a, b])\). If \(L : [a, b] \to \mathbb{R}\) is nonnegative, integrable and symmetric with respect to \(\frac{a+b}{2}\), then the following inequalities for fractional integrals hold:

\[
K \left( \frac{a+b}{2} \right) \left[ J^\alpha_{a+}L(b) + J^\alpha_{b-}L(a) \right] \leq \left[ J^\alpha_{a+}K(b) + J^\alpha_{b-}K(a) \right]
\]

with \(\alpha > 0\).

Let us now consider a bi-dimensional interval which will be used throughout this paper. So, we define \(\Delta = [a, b] \times [c, d]\) in \(\mathbb{R}^2\) with \(a < b\) and \(c < d\). A mapping \(\Phi : \Delta \to \mathbb{R}\) is said to be convex on the co-ordinates \(\Delta\) if the following inequality:

\[
\Phi(tx + (1-t)z, ty + (1-t)r) \leq t\Phi(x, y) + (1-t)\Phi(z, r)
\]

holds, for all \((x, y), (z, r) \in \Delta\) and \(t \in [0, 1]\).

A function \(\Phi : \Delta \to \mathbb{R}\) is said to be convex on the co-ordinates on \(\Delta\) if the partial mappings \(\Phi_y : [a, b] \to \mathbb{R}\), \(\Phi_y(u) = \Phi(u, y)\) and \(\Phi_x : [c, d] \to \mathbb{R}\), \(\Phi_x(v) = \Phi(x, v)\) are convex where defined for all \(x \in [a, b]\) and \(y \in [c, d]\) (see, [10]).

A formal definition for co-ordinated convex functions may be stated as follows:

**Definition 2.** (\([10]\)) A function \(\Phi : \Delta \to \mathbb{R}\) will be called co-ordinated convex on \(\Delta\), for all \(t, s \in [0, 1]\) and \((x, y), (u, r) \in \Delta\), if the following inequality holds:

\[
\Phi(tx + (1-t)u, sy + (1-s)r) \leq ts\Phi(x, u) + s(1-t)\Phi(y, u) + t(1-s)\Phi(x, r) + (1-t)(1-s)\Phi(y, r).
\]

Clearly, every convex function is a co-ordinated convex. Furthermore, there exists a co-ordinated convex function which is not convex, (see, [10]).

For several recent results concerning Hermite-Hadamard’s inequality for some convex function on the co-ordinates on a rectangle of \(\mathbb{R}^2\), we refer the reader to (\([1]-[3],[10],[15]-[17],[20],[22],[27]\)).

In [10], Dragomir established the following inequality of Hermite-Hadamard-type for co-ordinated convex mapping on a rectangle of \(\mathbb{R}^2\) similar to [14].

**Theorem 4.** Suppose that \(\Phi : \Delta \to \mathbb{R}\) is co-ordinated convex on \(\Delta\). Then one has the inequalities:
Theorem 5. Let \( \Phi : \Delta \to R \) be a co-ordinated convex function. Then the following double inequality hold:

\[
\Phi \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \\
\leq \frac{1}{2} \left[ \frac{1}{b - a} \int_a^b \Phi \left( x, \frac{c + d}{2} \right) \, dx + \frac{1}{d - c} \int_c^d \Phi \left( \frac{a + b}{2}, y \right) \, dy \right] \\
\leq \frac{1}{4} \left[ \frac{1}{b - a} \int_a^b \Phi (x, c) \, dx + \frac{1}{b - a} \int_a^b \Phi (x, d) \, dx \right. \\
\left. + \frac{1}{d - c} \int_c^d \Phi (a, y) \, dy + \frac{1}{d - c} \int_c^d \Phi (b, y) \, dy \right] \\
\leq \frac{\Phi (a, c) + \Phi (a, d) + \Phi (b, c) + \Phi (b, d)}{4}.
\]

The above inequalities are sharp.

Later, in [27], Sarikaya and Yaldiz proved inequalities of the Hermite-Hadamard type by using the definition of co-ordinated convex functions for L-Lipschitzian mappings. In [3], a Hermite-Hadamard-Fejer type inequality for co-ordinated convex mappings is established as follows:

**Theorem 5.** Let \( \Phi : \Delta \to R \) be a co-ordinated convex function. Then the following double inequality hold:

\[
\Phi \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \\
\leq \frac{1}{2} \left[ \frac{1}{b - a} \int_a^b \Phi \left( x, \frac{c + d}{2} \right) \, dx \right. \\
+ \left. \frac{1}{d - c} \int_c^d \Phi \left( \frac{a + b}{2}, y \right) \, dy \right] \\
\leq \frac{1}{4} \left[ \frac{1}{b - a} \int_a^b \Phi (x, c) \, dx + \frac{1}{b - a} \int_a^b \Phi (x, d) \, dx \right. \\
\left. + \frac{1}{d - c} \int_c^d \Phi (a, y) \, dy + \frac{1}{d - c} \int_c^d \Phi (b, y) \, dy \right] \\
\leq \frac{\Phi (a, c) + \Phi (a, d) + \Phi (b, c) + \Phi (b, d)}{4}.
\]

The above inequalities are sharp.

Because of the wide application of Hermite Hadamard type inequalities, Fejer type inequalities and Riemann-Liouville integrals for two-variable functions, many authors extend their studies to Hermite Hadamard type inequalities and Fejer type inequalities involving Riemann-Liouville integrals not limited to integer integrals.

**Definition 3.** (12, 18) Let \( \Phi \in L_1 (\Delta) \).

The Rieman-Liouville integrals \( J_a^{\alpha, \beta} \) of order \( \alpha, \beta > 0 \) with \( a, c \geq 0 \) are defined by

\[
J_a^{\alpha, \beta} \Phi (x, y) = \\
\frac{1}{\Gamma (\alpha) \Gamma (\beta)} \int_a^b \int_c^d (x - t)^{\alpha - 1} (y - s)^{\beta - 1} \Phi (t, s) \, ds \, dt,
\]

where \( p : \Delta \to R \) is positive, integrable and symmetric with respect to \( x = \frac{a + b}{2} \) and \( y = \frac{c + d}{2} \) on the co-ordinates on \( \Delta \). The above inequalities are sharp.
by using two different methods. We begin by the following theorem:

**Theorem 7.** Let \( \Phi : \Delta \to \mathbb{R} \) be a co-ordinated convex function such that \( \Phi \in L_1(\Delta) \). If \( \Psi : \Delta \to \mathbb{R} \) is nonnegative, integrable and symmetric with respect to \( x+b, c+d \) on the co-ordinates, then for any \( \alpha, \beta > 0 \) with \( a, c \geq 0 \), the following integral inequalities hold

\[
\Phi\left(\frac{a+b + c+d}{2}\right) \leq \frac{1}{4} \left[ J_{a+c+}^{\alpha,\beta} \Phi (b, d) + J_{a+d-}^{\alpha,\beta} \Phi (b, c) + J_{b-c-}^{\alpha,\beta} \Psi (a, d) + J_{b-d-}^{\alpha,\beta} \Psi (a, c) \right] \leq J_{b-c-}^{\alpha,\beta} \Psi (a, d) + J_{b-d-}^{\alpha,\beta} \Psi (a, c)
\]

**Proof.** Since \( \Phi \) is a convex function on \( \Delta \), then for all \( (t, s) \in [0, 1] \times [0, 1] \), we can write:

\[
\Phi\left(\frac{a+b + c+d}{2}\right) = \Phi\left(\frac{ta + (1-t)b + (1-t)b\cdot a + tb}{2}, \frac{sc + (1-s)d + (1-s)c + sd}{2}\right) \leq \frac{1}{4} \left[ \Phi (ta + (1-t)b, sc + (1-s)d) + \Phi (ta + (1-t)b, (1-s)c + sd) + \Phi ((1-t)a + tb, sc + (1-s)d) + \Phi ((1-t)a + tb, (1-s)c + sd) \right].
\]

Multiplying both sides of (9) by \( t^{n-1} s^{\beta-1} \Psi ((1-t)a + tb, (1-s)c + sd) \), and integrating the resulting inequality with respect to \( (t, s) \) on \([0, 1] \times [0, 1]\), we obtain

**2. Hermite-Hadamard-Fejer type inequalities for fractional integrals**

In this section, using Riemann-Liouville fractional integral operators, we establish new results on Hermite-Hadamard-Fejer type inequalities for co-ordinated convex functions. We present evidence...
\[
\Phi \left( \frac{a + b}{2}, \frac{c + d}{2} \right)
\times \int_0^1 \int_0^1 t^{a-1} s^{b-1} \Phi \left( (1-t)a + tb, (1-s)c + sd \right) ds dt \\
\leq \frac{1}{4} \int_0^1 \int_0^1 t^{a-1} s^{b-1} \left[ \Phi \left( ta + (1-t)b, sc + (1-s)d \right) + \Phi \left( (1-t)a + tb, sc + (1-s)d \right) \\
+ \Phi \left( (1-t)a + tb, (1-s)c + sd \right) \right] ds dt \\
\times \Psi \left( (1-t)a + tb, (1-s)c + sd \right) ds dt \\
+ \int_0^1 \int_0^1 t^{a-1} s^{b-1} \Phi \left( ta + (1-t)b, (1-s)c + sd \right) ds dt \\
\times \Psi \left( (1-t)a + tb, (1-s)c + sd \right) ds dt \\
+ \int_0^1 \int_0^1 t^{a-1} s^{b-1} \Phi \left( (1-t)a + tb, (1-s)c + sd \right) ds dt \\
\times \Psi \left( (1-t)a + tb, (1-s)c + sd \right) ds dt.
\]

\[
\frac{1}{(b-a)^{a-1}(d-c)^{b-1}} \Phi \left( \frac{a + b}{2}, \frac{c + d}{2} \right)
\times \int_a^b \int_c^d (x - a)^{a-1} (y - c)^{b-1} \Psi (x,y) dy dx \\
\leq \frac{1}{4 (b-a)^{a}(d-c)^{b}} \\
\times \left\{ \int_a^b \int_c^d (x - a)^{a-1} (y - c)^{b-1} \\
\times \Phi (a + b - x, c + d - y) \Psi (x,y) dy dx \\
+ \int_a^b \int_c^d (x - a)^{a-1} (d - y)^{b-1} \\
\times \Phi (a + b - x, y) \Psi (x,y) dy dx \\
+ \int_a^b \int_c^d (b - x)^{a-1} (y - c)^{b-1} \\
\times \Phi (x, c + d - y) \Psi (a + b - x, y) dy dx \\
+ \int_a^b \int_c^d (b - x)^{a-1} (d - y)^{b-1} \Phi (x,y) \Psi (x,y) dy dx \right\}
\]

Setting \( x = tb + (1-t)a \), \( y = sd + (1-s)c \) and \( dx = (b-a)dt \), \( dy = (d-c)ds \), we obtain:
\[= \frac{1}{4(b-a)^\alpha(d-c)\beta} \left\{ \int_a^b \int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1} \times \Phi(x, c + d - y) \Psi(x, y) dy dx \right. \]
\[+ \int_a^b \int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1} \Phi(x, y) \Psi(x, y) dy dx \]
\[+ \int_a^b \int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1} \times \Phi(x, c + d - y) \Psi(x, y) dy dx \]
\[+ \int_a^b \int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1} \Phi(x, y) \Psi(x, y) dy dx \right\}. \]

Therefore,
\[
\frac{\Gamma(\alpha) \Gamma(\beta)}{(b-a)^\alpha(d-c)\beta} \Phi \left( \frac{a+b+c+d}{2} \right) 
\times \left[ J_{a+c}^{\alpha,\beta} (\Phi \Psi) (b, d) + J_{a+d}^{\alpha,\beta} (\Phi \Psi) (b, c) 
+ J_{b+c}^{\alpha,\beta} (\Phi \Psi) (a, d) + J_{b+d}^{\alpha,\beta} (\Phi \Psi) (a, c) \right] \leq
\frac{\Gamma(\alpha) \Gamma(\beta)}{4(b-a)^\alpha(d-c)\beta} 
\times \left[ J_{a+c}^{\alpha,\beta} (\Phi \Psi) (b, d) + J_{a+d}^{\alpha,\beta} (\Phi \Psi) (b, c) 
+ J_{b+c}^{\alpha,\beta} (\Phi \Psi) (a, d) + J_{b+d}^{\alpha,\beta} (\Phi \Psi) (a, c) \right].
\]

The first inequality of (8) is thus proved. We shall prove the second inequality of (8): Since \(f\) is a convex function on \(\Delta\), then, for all \((t, s) \in [0, 1] \times [0, 1]\), it yields
\[
\Phi(ta + (1-t)b, sc + (1-s)d)
+ \Phi((1-t)a + tb, (1-s)c + sd)
+ \Phi((1-t)a + tb, (1-s)c + sd) \tag{10}
\leq \Phi(a, c) + \Phi(b, c) + \Phi(a, d) + \Phi(b, d).
\]

Then, multiplying both sides of (10) by \(t^{\alpha-1}s^{\beta-1}\Phi(tb + (1-t)a, sd + (1-s)c)\) and integrating the resulting inequality with respect to \((t, s)\) over \([0,1] \times [0,1]\), we get
\[
\int_0^1 \int_0^1 t^{\alpha-1}s^{\beta-1} \Phi(tb + (1-t)a, sc + (1-s)d) 
+ \Phi((1-t)a + tb, (1-s)c + sd)
+ \Phi((1-t)a + tb, (1-s)c + sd) \leq \Phi(a, c) + \Phi(b, c) + \Phi(a, d) + \Phi(b, d) \]
\[
\times \left[ J_{a+c}^{\alpha,\beta} (\Phi \Psi) (b, d) + J_{a+d}^{\alpha,\beta} (\Phi \Psi) (b, c) 
+ J_{b+c}^{\alpha,\beta} (\Phi \Psi) (a, d) + J_{b+d}^{\alpha,\beta} (\Phi \Psi) (a, c) \right].
\]

That is,
\[
\frac{1}{4} \left[ J_{a+c}^{\alpha,\beta} (\Phi \Psi) (b, d) + J_{a+d}^{\alpha,\beta} (\Phi \Psi) (b, c) 
+ J_{b+c}^{\alpha,\beta} (\Phi \Psi) (a, d) + J_{b+d}^{\alpha,\beta} (\Phi \Psi) (a, c) \right] 
\times \left[ J_{a+c}^{\alpha,\beta} (\Phi \Psi) (b, d) + J_{a+d}^{\alpha,\beta} (\Phi \Psi) (b, c) 
+ J_{b+c}^{\alpha,\beta} (\Phi \Psi) (a, d) + J_{b+d}^{\alpha,\beta} (\Phi \Psi) (a, c) \right].
\]

The proof of Theorem 7 is thus achieved. \(\Box\)

Remark 1. In Theorem 7
(i) If we take \(\alpha = \beta = 1\), then the inequality (8) becomes the inequality (9) of Theorem 6.
(ii) If we take \(\Psi(x, y) = 1\), then (8) becomes (7) of Theorem 6.

We prove also the following result:

Theorem 8. Let \(\Phi : \Delta \subset \mathbb{R}^2 \rightarrow \mathbb{R}\) be a co-ordinated convex function on \(\Delta\), with \(a, c \geq 0\), \(\alpha, \beta > 0\) and \(\Phi \in L_1(\Delta)\). If \(\Psi : \Delta \rightarrow \mathbb{R}\) is non-negative, integrable and symmetric with respect to \(\frac{a+b}{2}\) and \(\frac{c+d}{2}\) on the co-ordinates, then we have:
\[
\Phi \left( \frac{a+b}{2}, \frac{c+d}{2} \right) \times \left[ J_{a+c}^{\alpha,\beta} (\Psi(b, d) + J_{a+d}^{\alpha,\beta} (\Psi(b, c) 
+ J_{b+c}^{\alpha,\beta} (\Psi(a, d) + J_{b+d}^{\alpha,\beta} (\Psi(a, c) \right) \tag{11}
\]
On the Hermite-Hadamard-Fejer-type inequalities for co-ordinated convex functions ... 

\[ J_{a+}^\alpha \left[ \Phi \left( b, c + \frac{d}{2} \right) J_{c+}^\beta \Psi(b, d) \right] 
+ J_{a+}^\alpha \left[ \Phi \left( b, c + \frac{d}{2} \right) J_{c+}^\beta \Psi(b, c) \right] 
+ J_{c+}^\beta \left[ \Phi \left( a, c + \frac{d}{2} \right) J_{c-}^\alpha \Psi(a, d) \right] 
+ J_{c+}^\beta \left[ \Phi \left( a, c + \frac{d}{2} \right) J_{c-}^\alpha \Psi(a, c) \right] 
+ J_{c-}^\beta \left[ \Phi \left( a, c + \frac{d}{2} \right) J_{c-}^\alpha \Psi(a, c) \right] 
+ J_{c-}^\beta \left[ \Phi \left( a, c + \frac{d}{2} \right) J_{c-}^\alpha \Psi(a, c) \right] 
+ J_{d-}^\beta \left[ \Phi \left( a, c + \frac{d}{2} \right) J_{c-}^\alpha \Psi(a, c) \right] 
+ J_{d-}^\beta \left[ \Phi \left( a, c + \frac{d}{2} \right) J_{c-}^\alpha \Psi(a, c) \right] 
\leq \frac{\Phi(a, c) + \Phi(a, d) + \Phi(b, c) + \Phi(b, d)}{4} 
\times \left[ J_{a+c+}^{\alpha,\beta} \Psi(b, d) + J_{a+d-}^{\alpha,\beta} \Psi(b, c) 
+ J_{b-c+}^{\alpha,\beta} \Psi(a, d) + J_{b-d-}^{\alpha,\beta} \Psi(a, c) \right].

**Proof.** Since \( \Phi : \Delta \rightarrow \mathbb{R} \) is convex on the co-ordinates, it follows that the mapping \( F_x : [c, d] \rightarrow \mathbb{R}, F_x(y) = \Phi(x, y), \) is convex on \([c, d]\) and the mapping \( G_x : [c, d] \rightarrow \mathbb{R}, G_x(y) = \Psi(x, y) \) is nonnegative, integrable and symmetric with respect to \( \frac{c+d}{2} \), for all \( x \in [a, b] \). Then, thanks to the inequalities \((\ref{12})\), we can write

\[ F_x \left( \frac{c+d}{2} \right) \left[ J_{c+}^\beta G_x(d) + J_{d-}^\beta G_x(c) \right] \]

\[ \leq J_{c+}^\beta \left( F_x G_x(d) \right) + J_{d-}^\beta \left( F_x G_x(c) \right) \]

\[ \leq \frac{F_x(c) + F_x(d)}{2} \left[ J_{c+}^\beta G_x(d) + J_{d-}^\beta G_x(c) \right], \]

\[ x \in [a, b]. \]

That is,

\[ \Phi \left( x, \frac{c+d}{2} \right) \frac{1}{\Gamma(\beta)} \left[ \int_c^d (d-y)^{\beta-1} \Psi(x, y) dy \right] 
+ \int_c^d (y-c)^{\beta-1} \Psi(x, y) dy \]

\[ \leq \frac{\Phi(x, c) + \Phi(x, d)}{2} \frac{1}{\Gamma(\beta)} \left[ \int_c^d (d-y)^{\beta-1} \Psi(x, y) dy \right] 
+ \int_c^d (y-c)^{\beta-1} \Psi(x, y) dy, \]

for all \( x \in [a, b] \).

Multiplying both sides of \((\ref{12})\) by \( \frac{(b-x)^{\alpha-1}}{\Gamma(\alpha)} \) and \( \frac{(x-a)^{\beta-1}}{\Gamma(\beta)} \), and integrating with respect to \( x \) over \([a, b]\), respectively, we have

\[ \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^b \int_c^d (b-x)^{\alpha-1}(d-y)^{\beta-1} 
\times \Phi \left( \frac{x}{2} \right) \Psi(x, y) dydx \]

\[ + \frac{1}{\Gamma(\alpha) \Gamma(\beta)} \int_a^b \int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1} 
\times \Phi \left( \frac{x}{2} \right) \Psi(x, y) dydx \]
\[
\leq \frac{1}{\Gamma(a) \Gamma(b)} \int_a^b \int_c^d (b-x)^{a-1}(d-y)^{b-1} \times \Phi (x, y) \Psi (x, y) dydx
\]
\[
+ \int_a^b \int_c^d (b-x)^{a-1}(y-c)^{\beta-1}\Phi (x, y) \Psi (x, y) dydx \]
\[
\leq \frac{1}{2\Gamma(a) \Gamma(b)} \left[ \int_a^b \int_c^d (b-x)^{a-1}(d-y)^{b-1} \times \Phi (x, c) \Psi (x, y) dydx + \int_a^b \int_c^d (b-x)^{a-1}(y-c)^{\beta-1}\Phi (x, c) \Psi (x, y) dydx \right] \tag{14}
\]
\[
\leq \frac{1}{\Gamma(a) \Gamma(b)} \times \left[ \int_a^b \int_c^d (x-a)^{a-1}(d-y)^{b-1} \times \Phi (x, y) \Psi (x, y) dydx \right].
\]

For the mappings \( F_y : [a, b] \to \mathbb{R}, F_y(x) = \Phi(x, y) \) and \( G_y : [a, b] \to \mathbb{R}, G_y(x) = \Psi(x, y) \), we use the same arguments as before. So, we can state that
\[
\leq \frac{1}{\Gamma(a) \Gamma(b)} \int_a^b \int_c^d (b-x)^{a-1}(d-y)^{b-1} \times \Phi \left( \frac{a + b}{2}, y \right) \Psi (x, y) dydx \tag{16}
\]
\[
+ \frac{1}{\Gamma(a) \Gamma(b)} \int_a^b \int_c^d (x-a)^{a-1}(d-y)^{b-1} \times \Phi \left( \frac{a + b}{2}, y \right) \Psi (x, y) dydx
\]
\[
\leq \frac{1}{\Gamma(a) \Gamma(b)} \left[ \int_a^b \int_c^d (b-x)^{a-1}(d-y)^{b-1} \times \Phi (x, y) \Psi (x, y) dydx \right]
\]
\[
+ \int_a^b \int_c^d (x-a)^{a-1}(d-y)^{b-1} \times \Phi (x, y) \Psi (x, y) dydx \tag{17}
\]
\[
+ \int_a^b \int_c^d (b-x)^{a-1}(y-c)^{\beta-1}\Phi (x, y) \Psi (x, y) dydx
\]
\[
+ \int_a^b \int_c^d (x-a)^{a-1}(d-y)^{b-1}\Phi (x, d) \Psi (x, y) dydx
\]
\[
+ \int_a^b \int_c^d (b-x)^{a-1}(y-c)^{\beta-1}\Phi (x, c) \Psi (x, y) dydx
\].
Adding the inequalities \((13)-(17)\), we can write
\[
2 \Phi \left( x, y \right) \Psi \left( x, y \right) dydx 
+ \int_a^b \int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1} \Phi (a,y) \Psi (x,y) dydx 
+ \int_a^b \int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1} \Phi (b,y) \Psi (x,y) dydx 
+ \int_a^b \int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1} x \Phi (b,y) \Psi (x,y) dydx.
\]
These give the second and the third inequalities in \((11)\).

Now, by using the first inequality in \((3)\), it yields that
\[
\Phi \left( \frac{a+b}{2} \right) \times \left[ \int_a^b \int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1} \Phi (x,y) dydx \right] 
+ \int_a^b \int_c^d (x-a)^{\alpha-1}(y-c)^{\beta-1} \Psi (x,y) dydx 
\leq \int_a^b \int_c^d (b-x)^{\alpha-1}(y-c)^{\beta-1} x \Phi (x,y) dydx.
\]
Finally, by using the second inequality in (4), we can state that:

\[
\Phi \left( \frac{a + b}{2}, \frac{c + d}{2} \right) \times \left[ J_{a,c}^\alpha \Phi (b, d) + J_{a,d}^\alpha \Phi (b, c) + J_{b,c}^\alpha \Phi (a, d) + J_{b,d}^\alpha \Phi (a, c) \right] \\
\leq \frac{\Phi (a, d) + \Phi (b, d)}{2} \\
\leq \frac{\beta}{2 (d - c)^3} \left[ \int_c^d (d - y)^{\beta - 1} \Phi (a, y) \ dy + \int_c^d (y - c)^{\beta - 1} \Phi (b, y) \ dy \right] \\
\leq \frac{\Phi (a, c) + \Phi (a, d)}{2} \\
\leq \frac{\Phi (b, c) + \Phi (b, d)}{2}.
\]

By addition, we get the last inequality in (11).

Remark 2. In Theorem 8, if we take \( \alpha = \beta = 1 \), then the inequalities (11) become (7).

3. Conclusion

In this paper, we established the Hermite-Hadamard-Fejer type inequalities for co-ordinated mappings related results to present new type of inequalities involving Riemann-Liouville integral operator. The results presented in this paper would provide generalizations of those given in earlier works. The findings of this study have several significant implications for future applications.

References

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