New extensions of Chebyshev type inequalities using generalized Katugampola integrals via Pólya-Szegö inequality

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1. Introduction and preliminaries

This article is based on the well known Chebyshev functional [1]:

\[
T(f, g) = \frac{1}{b-a} \int_a^b f(x) g(x) \, dx - \left( \frac{1}{b-a} \int_a^b f(x) \, dx \right) \left( \frac{1}{b-a} \int_a^b g(x) \, dx \right),
\]

where \( f \) and \( g \) are two integrable functions which are synchronous on \([a, b]\), i.e.

\[
(f(x) - f(y))(g(x) - g(y)) \geq 0
\]

for any \( x, y \in [a, b] \). Then the Chebyshev inequality states that \( T(f, g) \geq 0 \).

For some recent counterparts, generalizations of Chebyshev inequality, the reader is refer to [2–6].

We also need to introduce the Pólya and Szegö inequality [7]:

\[
\frac{\int_a^b f^2(x) \, dx \int_a^b g^2(x) \, dx}{\left(\int_a^b f(x) g(x) \, dx\right)^2} \leq \frac{1}{4} \left( \sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}} \right)^2.
\]

Using the above Pólya-Szegö inequality, Dragomir and Diamond [8] established the following Grüss type inequality:

**Theorem 1.** Let \( f, g : [a, b] \to \mathbb{R}_+ \) be two integrable functions so that

\[0 < m \leq f(x) \leq M < \infty\]

and

\[0 < n \leq g(x) \leq N < \infty\]

for some \( a, b, M, N \in \mathbb{R}_+ \) with \( a < b \), then

\[
\frac{1}{b-a} \int_a^b f(x) \, dx \int_a^b g(x) \, dx \leq \frac{1}{4} \left( \sqrt{\frac{MN}{mn}} + \sqrt{\frac{mn}{MN}} \right)^2.
\]

For some recent counterparts, generalizations of Chebyshev inequality, the reader is refer to [2–6].

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for a.e. \( x \in [a, b] \).

Then, we have

\[
|T(f, g; a, b)| \leq \frac{1}{4} \frac{(M - m)(N - n)}{\sqrt{mnMN}} + \frac{1}{b - a} \int_a^b f(x)dx \frac{1}{b - a} \int_a^b g(x)dx.
\]

(1)

The constant \( \frac{1}{4} \) is best possible in (1) in the sense it cannot be replaced by a smaller constant.

For our purpose, we recall some other preliminaries: We note that the beta function \( B(\alpha, \beta) \) is defined by (see, e.g. [9 Section 1.1])

\[
B(\alpha, \beta) = \begin{cases} 
\int_0^1 t^{\alpha - 1}(1 - t)^{\beta - 1} dt & (\Re(\alpha) > 0; \\
\Gamma(\alpha) \Gamma(\beta) & (\alpha, \beta \in \mathbb{C} \setminus \mathbb{Z}_0^-),
\end{cases}
\]

where \( \Gamma \) is the familiar Gamma function. Here and in the following, let \( \mathbb{C}, \mathbb{R}, \mathbb{R}^+ \) and \( \mathbb{Z}_0^- \) be the sets of complex numbers, real numbers, positive real numbers and non-positive integers, respectively, and let \( \mathbb{R}^+_0 := \mathbb{R}^+ \cup \{0\} \).

**Definition 1.** (see, e.g., [10], [11]) Let \( [a, b] \) \((-\infty < a < b < \infty\) be a finite interval on the real axis \( \mathbb{R} \). The Riemann-Liouville fractional integrals (left-sided) of order \( \alpha \in \mathbb{C}, \Re(\alpha) > 0 \) of a real function \( f \in L(a, b) \), is defined:

\[
(I_{a+}^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x f(t) \frac{dt}{(x-t)^{1-\alpha}} \quad (x > a).
\]

(3)

**Definition 2.** (see, e.g., [10], [11]) Let \( (0 \leq a < b \leq \infty) \) be a finite or infinite interval on the half-axis \( \mathbb{R}^+ \). The Hadamard fractional integrals (left-sided) of order \( \alpha \in \mathbb{C}, \Re(\alpha) > 0 \) of a real function \( f \in L(a, b) \) are defined by

\[
(H_{a+}^\alpha f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x (\log \frac{x}{t})^{\alpha - 1} f(t) \frac{dt}{t} \quad (a < x < b).
\]

(4)

**Definition 3.** (see, e.g., [10], [11]) Let \( (\infty \leq -a < b \leq \infty) \) be a finite or infinite interval on the half-axis \( \mathbb{R}^+ \). Also let \( \Re(\alpha) > 0, \sigma > 0 \) and \( \eta \in \mathbb{C} \). The Erdelyi-Kober fractional integrals (left-sided) of order \( \alpha \in \mathbb{C} \) of a real function \( f \in L(a, b) \) are defined by

\[
(I_{a+}^{\alpha, \sigma, \eta} f)(x) = \frac{\sigma x^{-\sigma(\alpha+n)}}{\Gamma(\alpha)} \int_a^x t^{\sigma(\eta+1)-1} \frac{f(t)}{(x-t)^{1-\alpha}} dt \quad (0 \leq a < x < b \leq \infty).
\]

(5)

**Definition 4.** (see, e.g., [12]) Let \( [a, b] \subset \mathbb{R} \) be a finite interval. The Katugampola fractional integrals (left-sided) of order \( \alpha \in \mathbb{C}, \rho > 0 \Re(\alpha) > 0 \) of a real function \( f \in X^p(a, b) \) are defined by

\[
(I_{a+}^\alpha f)(x) := \frac{1}{\Gamma(\rho)} \int_{-\infty}^x (x-t)^{\rho-1} f(t) dt. \quad (x > a)
\]

(6)

**Definition 5.** (see, e.g., [12], [13]) Let a continuous function by parts in \( \mathbb{R} = (-\infty, \infty) \). The Liouville fractional integrals (left-sided) of order \( \alpha \in \mathbb{C}, \Re(\alpha) > 0 \) of a real function \( f \), are defined by

\[
(I_{a+}^\alpha f)(x) := \frac{1}{\Gamma(\rho)} \int_{-\infty}^x (x-t)^{\rho-1} f(t) dt. \quad (x \in \mathbb{R}).
\]

(7)

Here, the space \( X_p^\alpha(a, b) \) \((c \in \mathbb{R}, 1 \leq p \leq \infty) \) consists of those complex-valued Lebesgue measurable functions \( \varphi \) on \( (a, b) \) for which \( \|\varphi\|_{X_p^\alpha} < \infty \), with

\[
\|\varphi\|_{X_p^\alpha} = \left( \int_a^b |x^c \varphi(x)|^p dx \right)^{1/p} \quad (1 \leq p < \infty)
\]

and

\[
\|\varphi\|_{X_p^\infty} = \text{esssup}_{x \in (a, b)} |x^c \varphi(x)|
\]

In particular, when \( c = 1/p \) \((1 \leq p < \infty) \), the space \( X_p^\alpha(a, b) \) coincides with the classical \( L^p(a, b) \) space.

Let \( 0 \leq a < x < b < \infty \). Also, let \( \varphi \in X_p^\alpha(a, b), \alpha, \rho \in \mathbb{R}^+, \) and \( \beta, \eta, \kappa \in \mathbb{R} \). Then, the fractional integrals (left-sided and right-sided) of a function \( \varphi \) are defined, respectively, by (see [13])

\[
(I_{a+}^{\alpha, \beta, \eta, \kappa} \varphi)(x) := \frac{\rho^{1-\beta} x^{\kappa}}{\Gamma(\rho)} \int_a^x \frac{\tau^{\rho(\eta+1)-1} \varphi(\tau)}{(x^\rho - \tau^\rho)^{1-\alpha}} d\tau
\]

and

\[
(I_{a+}^{\alpha, \beta, \eta, \kappa} \varphi)(x) := \frac{\rho^{1-\beta} x^{\kappa}}{\Gamma(\rho)} \int_a^x \frac{\tau^{\rho(\eta+1)-1} \varphi(\tau)}{(x^\rho - \tau^\rho)^{1-\alpha}} d\tau
\]
The fractional integral $(\rho I_{b^{-,\eta,\kappa}}^\psi \varphi)(x)$ (see \[15, Eq. (3.1)\])
\[
:= \rho^{1-\beta} x^\rho \frac{\Gamma(\alpha)}{\Gamma(\alpha + \rho)} \int_x^b \frac{\tau^{\kappa+\rho-1}}{(\tau^\rho - x^\rho)^{1-\alpha}} \varphi(\tau) d\tau.
\]

Remark 1. The fractional integral in $(\mathcal{S})$ contains five well-known fractional integrals as its particular cases (see also $(\mathcal{J})$)

\begin{enumerate}
\item Setting $\kappa = 0$, $\eta = 0$ and $\rho = 1$ in $(\mathcal{S})$, the integral operator $(\mathcal{S})$ reduces to the Riemann-Liouville fractional integral $(\mathcal{B})$ (see also [10] p. 69).
\item Setting $\kappa = 0$, $\eta = 0$, $a = -\infty$ and $\rho = 1$ in $(\mathcal{S})$, the integral operator $(\mathcal{S})$ reduces to the Liouville fractional integral $(\mathcal{L})$ (see also [10] p. 79).
\item Setting $\beta = \alpha$, $\kappa = 0$, $\eta = 0$, and taking the limit $\rho \to 0^+$ with L’Hôpital’s rule in $(\mathcal{S})$, the integral operator $(\mathcal{S})$ reduces to the Erdélyi-Kober fractional integral $(\mathcal{E})$ (see also [10] p. 105).
\item Setting $\beta = 0$ and $\kappa = -\rho(\alpha + \eta)$ in $(\mathcal{S})$, the integral operator $(\mathcal{S})$ reduces to the Erdélyi-Kober fractional integral $(\mathcal{E})$ (see also [10] p. 105).
\item Setting $\beta = \alpha$, $\kappa = 0$ and $\eta = 0$ in $(\mathcal{S})$, the integral operator $(\mathcal{S})$ reduces to the Katugampola fractional integral $(\mathcal{K})$ (see also [12]).
\end{enumerate}

The principle aim of the present paper is to establish new Pólya-Szegö inequalities and other of Chebyshev type by using generalized Katugampola fractional integration theory.

2. Main Results

In this section, we establish some new Chebyshev type inequalities involving the Katugampola fractional integration approach. Thanks to $(\mathcal{S})$, we obtain (see [13] Eq. (3.1))
\[
(\rho I_{0+}^{\alpha, \beta, \eta, \kappa} \varphi)(x) := (\rho I_{0+}^{\alpha, \beta} \varphi)(x).
\]

Lemma 1. Let $\beta, \kappa \in \mathbb{R}$, $x, \alpha, \rho \in \mathbb{R}^+$, and $\eta \in \mathbb{R}_0^+$. Let $f$ and $g$ be two positive integrable functions on $(0, \infty)$. Assume that there exist four positive integrable functions $v_1$, $v_2$, $\omega_1$ and $\omega_2$, such that:
\[
0 < v_1(\tau) \leq f(\tau) \leq v_2(\tau)
\]
\[
0 < \omega_1(\tau) \leq g(\tau) \leq \omega_2(\tau)
\]
\[
(\tau \in [0, x], x > 0)
\]

Then the following inequality holds:
\[
\rho I_{0+}^{\alpha, \beta} \{v_1 \omega_1 f^2\} (x) \leq \frac{1}{4} \frac{v_2^2(x)}{\omega_2^2(x)}.
\]

\textbf{Proof.} From $(\mathcal{S})$, for $\tau \in [0, x]$, $x > 0$, we can write
\[
\left( \frac{v_2(\tau)}{\omega_1(\tau)} - \frac{f(\tau)}{g(\tau)} \right) \geq 0
\]
and
\[
\left( \frac{f(\tau)}{g(\tau)} - \frac{v_1(\tau)}{\omega_2(\tau)} \right) \geq 0.
\]

Multiplying $(\mathcal{L})$ and $(\mathcal{J})$, we get
\[
\left( \frac{v_2(\tau)}{\omega_1(\tau)} - \frac{f(\tau)}{g(\tau)} \right) \left( \frac{f(\tau)}{g(\tau)} - \frac{v_1(\tau)}{\omega_2(\tau)} \right) \geq 0.
\]

From the above inequality, we can write
\[
(v_1(\tau) \omega_1(\tau) + v_2(\tau) \omega_2(\tau)) f(\tau) g(\tau)
\geq \omega_1(\tau) \omega_2(\tau) f^2(\tau) + v_1(\tau) v_2(\tau) g^2(\tau).
\]

Multiplying both sides of $(\mathcal{S})$ by
\[
\rho I_{0+}^{\alpha, \beta} \{v_1 \omega_1 f^2\} (x) \leq \frac{1}{4} \frac{v_2^2(x)}{\omega_2^2(x)}
\]
and integrating the resulting inequality with respect to $\tau$ over $(0, x)$, we get
\[
0 \leq \rho I_{0+}^{\alpha, \beta} \{\omega_1 \omega_2 f^2\} (x) + \rho I_{0+}^{\alpha, \beta} \{v_1 v_2 g^2\} (x)
\]
Applying the AM-GM inequality, i.e. \( (a + b \geq 2\sqrt{ab}, \ a, b \in \mathbb{R}^+ ) \), we have
\[
\rho I_{\eta, \kappa}^{\alpha, \beta} \{ (v_1 \omega_1 + v_2 \omega_2) f g \} (x) \\
\geq 2\sqrt{\rho I_{\eta, \kappa}^{\alpha, \beta} \{ \omega_1 \omega_2 f^2 \} \rho I_{\eta, \kappa}^{\alpha, \beta} \{ v_1 v_2 g^2 \} (x)} \\
\]
which implies that
\[
\rho I_{\eta, \kappa}^{\alpha, \beta} \{ \omega_1 \omega_2 f^2 \} (x) \rho I_{\eta, \kappa}^{\alpha, \beta} \{ v_1 v_2 g^2 \} (x) \\
\leq \frac{1}{4} \left( \rho I_{\eta, \kappa}^{\alpha, \beta} \{ (v_1 \omega_1 + v_2 \omega_2) f g \} (x) \right)^2. 
\]
So, we get the desired result. \( \square \)

**Corollary 1.** If \( v_1 = m, \ v_2 = M, \ \omega_1 = n \) and \( \omega_2 = N \), then we have
\[
\left( \frac{\rho I_{\eta, \kappa}^{\alpha, \beta} f^2 (x) \rho I_{\eta, \kappa}^{\alpha, \beta} g^2 (x)}{(\rho I_{\eta, \kappa}^{\alpha, \beta} f g (x))^2} \right) \\
\leq \frac{1}{4} \left( \sqrt{\frac{mn}{MN}} + \sqrt{\frac{MN}{mn}} \right)^2. 
\]

**Remark 2.** Setting \( \kappa = 0, \ \eta = 0 \) and \( \rho = 1 \) in Lemma 1 yields the inequality in [10, Lemma 3.1].

**Lemma 2.** Let \( \beta, \kappa \in \mathbb{R}, \ x, \alpha, \theta, \rho \in \mathbb{R}^+, \) and \( \eta \in \mathbb{R}_+^+ \). Let \( f \) and \( g \) be two positive integrable functions on \( [0, \infty) \). Assume that there exist four positive integrable functions \( v_1, v_2, \omega_1 \) and \( \omega_2 \) satisfying condition [17]. Then the following inequality holds:
\[
\rho I_{\eta, \kappa}^{\alpha, \beta} \{ v_1 v_2 \} (x) \rho I_{\eta, \kappa}^{\alpha, \beta} \{ \omega_1 \omega_2 \} (x) \\
\times \rho I_{\eta, \kappa}^{\alpha, \beta} \{ f^2 \} (x) \rho I_{\eta, \kappa}^{\alpha, \beta} \{ g^2 \} (x) \\
\leq \frac{1}{4} \left( \rho I_{\eta, \kappa}^{\alpha, \beta} \{ v_1 f \} (x) \rho I_{\eta, \kappa}^{\alpha, \beta} \{ \omega_1 g \} (x) \right) \\
+ \rho I_{\eta, \kappa}^{\alpha, \beta} \{ v_2 f \} (x) \rho I_{\eta, \kappa}^{\alpha, \beta} \{ v_2 g \} (x) \right)^2. 
\]

**Proof.** From [11], we get
\[
\left( \frac{v_2 (\tau)}{\omega_1 (\xi)} - \frac{f(\tau)}{g(\xi)} \right) \geq 0 \\
\text{and} \\
\left( \frac{f(\tau)}{g(\xi)} - \frac{v_1 (\tau)}{\omega_2 (\xi)} \right) \geq 0
\]
which lead to
\[
\left( \frac{v_1 (\tau)}{\omega_2 (\xi)} + \frac{v_2 (\tau)}{\omega_1 (\xi)} \right) f(\tau) \\
\geq \frac{f^2 (\tau)}{g^2 (\xi)} + \frac{v_1 (\tau)v_2 (\tau)}{\omega_1 (\xi)\omega_2 (\xi)} \right)^2. 
\]

Multiplying both sides of (17) by \( \omega_1 (\xi)\omega_2 (\xi) g^2 (\xi) \), we get
\[
v_1 (\tau) f(\tau) \omega_1 (\xi) g(\xi) + v_2 (\tau) f(\tau) \omega_2 (\xi) g(\xi) \\
\geq \omega_1 (\xi)\omega_2 (\xi) f^2 (\tau) + v_1 (\tau)v_2 (\tau) g^2 (\xi). \]

Multiplying both sides of (18) by
\[
\rho^{2(1-\rho)x^2 \kappa} \tau^{\rho(\eta+1)-1} \eta^{\rho(\eta+1)-1} \xi^{\rho(\eta+1)-1} \\
\Gamma (\alpha) \Gamma (\theta) \left( x^\rho - \tau^\rho \right)^{-\alpha} \left( \rho - \xi \right)^{-\theta} 
\]
and integrating the resulting inequality with respect to \( \tau \) and \( \xi \) over \( (0, x)^2 \), we get
\[
\rho I_{\eta, \kappa}^{\alpha, \beta} \{ v_1 f \} (x) \rho I_{\eta, \kappa}^{\alpha, \beta} \{ \omega_1 g \} (x) \\
+ \rho I_{\eta, \kappa}^{\alpha, \beta} \{ v_2 f \} (x) \rho I_{\eta, \kappa}^{\alpha, \beta} \{ v_2 g \} (x) \\
\geq \rho I_{\eta, \kappa}^{\alpha, \beta} \{ f^2 \} (x) \rho I_{\eta, \kappa}^{\alpha, \beta} \{ \omega_1 \omega_2 \} (x) \\
+ \rho I_{\eta, \kappa}^{\alpha, \beta} \{ v_1 v_2 \} (x) \rho I_{\eta, \kappa}^{\alpha, \beta} \{ g^2 \} (x). 
\]

Applying the AM-GM inequality, we have
\[
\rho I_{\eta, \kappa}^{\alpha, \beta} \{ v_1 v_2 \} (x) \rho I_{\eta, \kappa}^{\alpha, \beta} \{ \omega_1 \omega_2 \} (x) \\
\times \rho I_{\eta, \kappa}^{\alpha, \beta} \{ f^2 \} (x) \rho I_{\eta, \kappa}^{\alpha, \beta} \{ g^2 \} (x) \\
\geq 2 \sqrt{\rho I_{\eta, \kappa}^{\alpha, \beta} \{ f^2 \} (x) \rho I_{\eta, \kappa}^{\alpha, \beta} \{ \omega_1 \omega_2 \} (x)} \\
\times \sqrt{\rho I_{\eta, \kappa}^{\alpha, \beta} \{ v_1 v_2 \} (x) \rho I_{\eta, \kappa}^{\alpha, \beta} \{ g^2 \} (x). 
\]

So, we get the desired inequality of (16). \( \square \)

**Corollary 2.** If \( v_1 = m, \ v_2 = M, \ \omega_1 = n \) and \( \omega_2 = N \), then we have
\[
\Lambda_{\alpha, \beta}^{\rho, \eta} (\alpha, \eta) \Lambda_{\alpha, \beta}^{\rho, \eta} (\beta, \eta) \\
\times \left( \rho I_{\eta, \kappa}^{\alpha, \beta} f^2 (x) \rho I_{\eta, \kappa}^{\alpha, \beta} g^2 (x) \right) \left( \rho I_{\eta, \kappa}^{\alpha, \beta} f (x) \rho I_{\eta, \kappa}^{\alpha, \beta} g (x) \right)^2 \\
\leq \frac{1}{4} \left( \sqrt{\frac{mn}{MN}} + \sqrt{\frac{MN}{mn}} \right)^2. 
\]

**Remark 3.** Setting \( \kappa = 0, \ \eta = 0 \) and \( \rho = 1 \) in Lemma[2] yields the inequality in [10, Lemma 3.3].
Lemma 3. Suppose that all assumptions of Lemma 2 are satisfied. Then, we have:

\[
\rho I_{\eta, \kappa}^{\alpha, \beta} \{f^2\}(x) \leq \rho I_{\eta, \kappa}^{\alpha, \beta} \{g^2\}(x)
\] (19)

\[
\rho I_{\eta, \kappa}^{\alpha, \beta} \{f^2\}(x) \leq \rho I_{\eta, \kappa}^{\alpha, \beta} \{g^2\}(x)\]

Proof. Using the condition (11), we get

\[
\frac{\rho^{1-\beta} x^\kappa}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} f^2(\tau) d\tau
\]

\[
\leq \frac{\rho^{1-\beta} x^\kappa}{\Gamma(\alpha)} \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \omega_1(\tau) g(\tau) d\tau
\]

which leads to

\[
\rho I_{\eta, \kappa}^{\alpha, \beta} \{f^2\}(x) \leq \rho I_{\eta, \kappa}^{\alpha, \beta} \{g^2\}(x).
\] (20)

Similarly, we have

\[
\frac{\rho^{1-\beta} x^\kappa}{\Gamma(\theta)} \int_0^x \frac{\xi^{\rho(\eta+1)-1}}{(x^\rho - \xi^\rho)^{1-\theta}} g^2(\xi) d\xi
\]

\[
\leq \frac{\rho^{1-\beta} x^\kappa}{\Gamma(\theta)} \int_0^x \frac{\omega_2(\xi)}{(x^\rho - \xi^\rho)^{1-\theta}} \omega_1(\xi) f(\xi) g(\xi) d\xi
\]

which implies

\[
\rho I_{\eta, \kappa}^{\alpha, \beta} \{g^2\}(x) \leq \rho I_{\eta, \kappa}^{\alpha, \beta} \{f^2\}(x).
\] (21)

Multiplying (20) and (21), we get the inequality of (19).

\[
|G_1(f, v_1, v_2)(x) + G_2(f, v_1, v_2)(x)|^{1/2}
\]

\[
\leq |G_1(f, v_1, v_2)(x) + G_2(f, v_1, v_2)(x)|^{1/2}
\]

where

\[
G_1(f, v_1, v_2)(x) = \frac{\left(\rho I_{\eta, \kappa}^{\alpha, \beta} \{v_1 + v_2\} f(x)\right)^2}{\rho I_{\eta, \kappa}^{\alpha, \beta} \{v_1 v_2\}(x)}
\]

and

\[
G_2(f, v_1, v_2)(x) = \frac{\left(\rho I_{\eta, \kappa}^{\alpha, \beta} \{v_1 + v_2\} f(x)\right)^2}{\rho I_{\eta, \kappa}^{\alpha, \beta} \{v_1 v_2\}(x)}
\]

\[
\rho I_{\eta, \kappa}^{\alpha, \beta} \{f^2\}(x) \leq \rho I_{\eta, \kappa}^{\alpha, \beta} \{g^2\}(x).
\]

Corollary 3. If \( v_1 = m, v_2 = M, \omega_1 = n \) and \( \omega_2 = N \), then we have

\[
\frac{(\rho I_{\eta, \kappa}^{\alpha, \beta} f^2)(x)}{(\rho I_{\eta, \kappa}^{\alpha, \beta} f^2)(x)} \leq \frac{MN}{mn}.
\]

Remark 4. Setting \( \kappa = 0, \eta = 0 \) and \( \rho = 1 \) in Lemma 3 yields the inequality in [16, Lemma 3.4].

Theorem 2. Let \( \beta, \kappa, x, \alpha, \theta, \rho \in \mathbb{R}^+ \), and \( \eta \in \mathbb{R}^+_0 \). Let \( f \) and \( g \) be two positive integrable functions on \([0, \infty)\). Assume also that there exist four positive integrable functions \( v_1, v_2, \omega_1 \) and \( \omega_2 \) satisfying the condition (17). Then the following inequality holds:

\[
H(\tau, \xi) = (f(\tau) - f(\xi))(g(\tau) - g(\xi)),
\]

equivalently,

\[
H(\tau, \xi) = f(\tau)g(\xi) + f(\xi)g(\xi) - f(\tau)g(\xi) - f(\xi)g(\tau).
\] (23)

Proof. Let \( f \) and \( g \) be two positive integrable functions on \([0, \infty)\). For \( \tau, \xi \in (0, x) \) with \( x > 0 \), we define \( H(\tau, \xi) \) as

\[
H(\tau, \xi) = (f(\tau) - f(\xi))(g(\tau) - g(\xi)),
\]

Multiplying both sides of (23) by

\[
\frac{\rho^{2(1-\beta)} \tau^{\rho(\eta+1)-1}}{\Gamma(\alpha) \Gamma(\theta)} (x^\rho - \tau^\rho)^{1-\alpha} (x^\rho - \xi^\rho)^{1-\theta}
\]
and double integrating the resulting inequality with respect to \( \tau \) and \( \xi \) over \((0, x)^2\), we get

\[
\frac{\rho^2 (1 - \beta) x^{2\kappa}}{\Gamma(\alpha) \Gamma(\theta)} \int_0^x \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \times \frac{\xi^{\rho(\eta+1)-1}}{(x^\rho - \xi^\rho)^{1-\theta}} H(\tau, \xi) \, d\tau \, d\xi
\]

\[
= \Lambda_{x,\kappa}^\beta(\alpha, \eta) \left( \rho_1^{\rho,\beta} f g \right)(x)
+ \Lambda_{x,\kappa}^\beta(\theta, \eta) \left( \rho_1^{\rho,\beta} f g \right)(x)
- \left( \rho_1^{\rho,\beta} f \right)(x) \left( \rho_1^{\rho,\beta} g \right)(x)
- \left( \rho_1^{\rho,\beta} g \right)(x) \left( \rho_1^{\rho,\beta} f \right)(x).
\] (24)

Applying the Cauchy-Schwarz inequality for double integrals, we write

\[
\left| \frac{\rho^2 (1 - \beta) x^{2\kappa}}{\Gamma(\alpha) \Gamma(\theta)} \int_0^x \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \times \frac{\xi^{\rho(\eta+1)-1}}{(x^\rho - \xi^\rho)^{1-\theta}} H(\tau, \xi) \, d\tau \, d\xi \right|
\]

\[
\leq \frac{\rho^2 (1 - \beta) x^{2\kappa}}{\Gamma(\alpha) \Gamma(\theta)} \int_0^x \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \times \frac{\xi^{\rho(\eta+1)-1}}{(x^\rho - \xi^\rho)^{1-\theta}} H(\tau, \xi) \, d\tau \, d\xi
\]

\[
\leq \frac{\rho^2 (1 - \beta) x^{2\kappa}}{\Gamma(\alpha) \Gamma(\theta)} \int_0^x \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \times \frac{\xi^{\rho(\eta+1)-1}}{(x^\rho - \xi^\rho)^{1-\theta}} f^2(\tau) d\tau \, d\xi
\]

\[
- \frac{\rho^2 (1 - \beta) x^{2\kappa}}{\Gamma(\alpha) \Gamma(\theta)} \int_0^x \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \times \frac{\xi^{\rho(\eta+1)-1}}{(x^\rho - \xi^\rho)^{1-\theta}} f(\tau) f(\xi) d\tau \, d\xi \frac{1}{2}
\]

\[
\leq \frac{\rho^2 (1 - \beta) x^{2\kappa}}{\Gamma(\alpha) \Gamma(\theta)} \int_0^x \int_0^x \frac{\tau^{\rho(\eta+1)-1}}{(x^\rho - \tau^\rho)^{1-\alpha}} \times \frac{\xi^{\rho(\eta+1)-1}}{(x^\rho - \xi^\rho)^{1-\theta}} g(\tau) g(\xi) d\tau \, d\xi \frac{1}{2}
\]

Therefore,

![Equation (24)](image)

Applying Lemma [1] with \( \omega_1(\tau) = \omega_2(\tau) = g(\tau) = 1 \), we get

\[
\Lambda_{x,\kappa}^\beta(\theta, \eta) \left( \rho_1^{\rho,\beta} f^2 \right)(x)
\]

\[
\leq \frac{\Lambda_{x,\kappa}^\beta(\theta, \eta)}{4} \left( \rho_1^{\rho,\beta} \left\{ (v_1 + v_2) f \right\}(x) \right)^2.
\]

This implies that

\[
\Lambda_{x,\kappa}^\beta(\theta, \eta) \left( \rho_1^{\rho,\beta} f \right)(x)
\]

\[
- \left( \rho_1^{\rho,\beta} f \right)(x) \left( \rho_1^{\rho,\beta} f \right)(x)
\]

\[
\leq \frac{\Lambda_{x,\kappa}^\beta(\theta, \eta)}{4} \left( \rho_1^{\rho,\beta} \left\{ (v_1 + v_2) f \right\}(x) \right)^2
\]

\[
- \left( \rho_1^{\rho,\beta} f \right)(x) \left( \rho_1^{\rho,\beta} f \right)(x)
\]

\[
= G_1(f, v_1, v_2)(x).
\] (26)

and

\[
\Lambda_{x,\kappa}^\beta(\theta, \eta) \left( \rho_1^{\rho,\beta} f \right)(x)
\]

\[
- \left( \rho_1^{\rho,\beta} f \right)(x) \left( \rho_1^{\rho,\beta} f \right)(x)
\]

\[
\leq \frac{\Lambda_{x,\kappa}^\beta(\theta, \eta)}{4} \left( \rho_1^{\rho,\beta} \left\{ (\omega_1 + \omega_2) f \right\}(x) \right)^2
\]

\[
- \left( \rho_1^{\rho,\beta} f \right)(x) \left( \rho_1^{\rho,\beta} f \right)(x)
\]

\[
= G_2(f, \omega_1, \omega_2)(x).
\] (27)

Similarly, applying Lemma [1] with \( v_1(\tau) = v_2(\tau) = f(\tau) = 1 \), we have
Using (26)-(29), we conclude the desired result.

Remark 5. Setting $\kappa = 0$, $\eta = 0$ and $\rho = 1$ in Theorem 3 yields the inequality in (16, Theorem 3.7).

**Theorem 3.** Assume that all conditions of Theorem 2 are fulfilled. Then, we have:

\[
\begin{align*}
\left| \Lambda_{x,\kappa}^{\rho,\beta}(\alpha, \eta) \left( \rho I_{\eta,\kappa}^{\alpha,\beta} f \right) (x) \right| \\
- \left( \rho I_{\eta,\kappa}^{\alpha,\beta} f \right) (x) \left( \rho I_{\eta,\kappa}^{\alpha,\beta} g \right) (x) \\
\leq |G(f, v_1, v_2)(x)G(g, \omega_1, \omega_2)(x)|^{1/2}
\end{align*}
\]

(30)

where

\[
G(u, v, w)(x) = \frac{\Lambda_{x,\kappa}^{\rho,\beta}(\alpha, \eta)}{4}
\]

\[
\times \left( \rho I_{\eta,\kappa}^{\alpha,\beta} \{ (v + w)u \} (x) \right)^2
\]

\[
- \left( \rho I_{\eta,\kappa}^{\alpha,\beta} \{ vw \} (x) \right)^2.
\]

**Proof.** Setting $\alpha = \theta$ in (22), we obtain (30).

**Corollary 4.** If $v_1 = m$, $v_2 = M$, $\omega_1 = n$ and $\omega_2 = N$, then we have

\[
G(f, m, M)(x) = \frac{(M - m)^2}{4mm} \left( \rho I_{\eta,\kappa}^{\alpha,\beta} f \right)^2,
\]

\[
G(g, n, N)(x) = \frac{(N - n)^2}{4nn} \left( \rho I_{\eta,\kappa}^{\alpha,\beta} g \right)^2.
\]

**Remark 6.** We consider some particular cases of the result in Theorem 3.

(i) Setting $\kappa = 0$, $\eta = 0$ and $\rho = 1$ in the result in Theorem 3 yields the inequality in (16, Theorem 3.7).

(ii) Setting $\beta = 0$ and $\kappa = -\rho(\alpha + \eta)$ in the result in inequality (30) yields to

\[
\left| \frac{\Gamma(\eta + 1)}{\Gamma(\alpha + \eta + 1)} \left( I_{0+}^{\alpha,\rho,\eta} f \right) (x) \right|
\]

\[
- \left( I_{0+}^{\alpha,\rho,\eta} f \right) (x) \left( I_{0+}^{\alpha,\rho,\eta} g \right) (x)
\]

\[
\leq |G(f, v_1, v_2)(x)G(g, \omega_1, \omega_2)(x)|^{1/2}
\]

where

\[
G(u, v, w)(x) = \frac{\Gamma(\eta + 1)}{4\Gamma(\alpha + \eta + 1)}
\]

\[
\times \left( I_{0+}^{\alpha,\rho,\eta} \{ (v + w)u \} (x) \right)^2
\]

\[
- \left( I_{0+}^{\alpha,\rho,\eta} \{ vw \} (x) \right)^2.
\]

(iii) Setting $\beta = \alpha$, $\kappa = 0$ and $\eta = 0$ in the result in Theorem 3, under the corresponding reduced assumption, we obtain

\[
\left| \frac{x^{\rho\alpha}}{\Gamma(\alpha + 1)} \left( I_{0+}^{\alpha,\rho} f \right) (x) \right|
\]

\[
- \left( I_{0+}^{\alpha,\rho} f \right) (x) \left( I_{0+}^{\alpha,\rho} g \right) (x)
\]

\[
\leq |G(f, v_1, v_2)(x)G(g, \omega_1, \omega_2)(x)|^{1/2}
\]

where

\[
G(u, v, w)(x) = \frac{\Gamma(\alpha + 1)}{4\Gamma(\alpha + 1)}
\]

\[
\times \left( I_{0+}^{\alpha,\rho} \{ (v + w)u \} (x) \right)^2
\]

\[
- \left( I_{0+}^{\alpha,\rho} \{ vw \} (x) \right)^2.
\]

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