A Boiti-Leon Pimpinelli equations with time-conformable derivative

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ARTICLE INFO

Article History:
Received 02 January 2019
Accepted 27 July 2019
Available 13 September 2019

Keywords:
Sinh-Gordon expansion method
(2 + 1)-Boiti-Leon Pempinelli equations
Conformable derivative
Jacobi Elliptic functions

AMS Classfication 2010:
35Qxx; 26A33

1. Introduction

Partial differential equations play an important role in interpretation and modeling of many phenomena appearing in applied mathematics and physics including fluid mechanics, electrical circuits, diffusion, damping laws, relaxation processes, optimal control theory, solid mechanics, propagation of waves, chemistry, biology, and so on. Therefore, seeking solutions for partial differential equations is an important aspect of scientific research. Besides, many scientists have focused on new findings to the nonlinear partial differential equations, such as traveling wave solutions, complex functions, trigonometric functions, Jacobi elliptic functions, and so on. For constructing such solutions, there exist numerous efficient techniques. For example, Sumudu homotopy perturbation transform method \cite{1-4}, Lie symmetry method \cite{5}, tan(\phi(\xi)/2) – expansion method \cite{6,7}, generalized trigonometry functions \cite{8}, Riccati equation expansion technique \cite{9}, Jacobi elliptic function technique \cite{10}, etc. For more informations about the analytical methods, we refer the reader to the following references \cite{12-20}.

In this article, we adopt a transformation method based on a sinh-Gordon expansion equation to obtain new soliton solutions of Boiti-Leon Pimpinelli equations (BLP) with conformable derivative. For more details on BLP equation we refer the reader to the references \cite{21-23}.

On the other hand, the following equation
\[
\frac{\partial^2 u}{\partial x \partial t} = \alpha \sinh u,
\]
(1)
is called Sinh-Gordon equation and arises in various areas of nonlinear sciences, where \( \alpha \) is an arbitrary constant.

Using the traveling wave transformation
\[
\begin{cases}
  u(x, t) = U(\xi) \\
  \xi = \mu (x + y - \lambda t),
\end{cases}
\]
(2)
equation (1) is converted to
\[
\frac{\partial^2 U}{\partial \xi^2} = -\frac{\alpha}{\mu^2} \sinh U,
\]

where the coefficients \(\mu\) and \(\lambda\) stands for the wave number and wave speed, respectively. Now, integrating (3) yields to

\[
\left( \frac{d}{d \xi} \frac{1}{2} U \right)^2 = -\frac{\alpha}{\mu^2} \sinh^2 \left( \frac{1}{2} U \right) + c,
\]

where \(c\) is an integration constant. Consider the following

\[
c = 0, \quad \alpha = -\mu^2 \lambda \quad \text{and} \quad \frac{1}{2} U = w,
\]

equation (4) takes the form

\[
\left( \frac{dw}{d\xi} \right)^2 = \sinh^2 w + c,
\]

which can be used in the adopted method, where \(c\) is an integration constant. Consider the following

\[
dw(\xi) = \sinh w(\xi).
\]

To construct Jacobi elliptic function solutions, we convert equation (3) into the following

\[
\frac{d^2 w}{d\xi^2} = \frac{1}{2} \sinh 2w,
\]

under the assumptions \(\phi = 2w\) and \(-\frac{\alpha}{\mu^2} \lambda = 1\).

Equation (6) can be also written as

\[
\left( \frac{dw}{d\xi} \right)^2 = \sinh^2 w + c,
\]

which can be used in the adopted method, where \(c\) is an integration constant. Therefore, Equation (7) has the following solutions

\[
\sinh [w(\xi)] = \text{cs}(\xi; m),
\]

\[
\cosh [w(\xi)] = \text{ns}(\xi; m),
\]

where \(m\) is the modulus of the Jacobian elliptic functions:

\[
\text{cs}(\xi; m) = \frac{cn(\xi; m)}{sn(\xi; m)},
\]

\[
\text{ns}(\xi; m) = \frac{1}{sn(\xi; m)},
\]

with the properties

\[
\frac{d \text{cs}(\xi; m)}{d\xi} = -\text{ns}(\xi; m) \, d\xi(\xi; m),
\]

\[
\frac{d \text{ns}(\xi; m)}{d\xi} = -\text{cs}(\xi; m) \, d\xi(\xi; m).
\]

Substitution of (8) and (9) in (10) reveals that the constant \(c\) must satisfy

\[
c = 1 - m^2,
\]

which is used throughout this work.

The plan of this paper is as follows: In section 2 some properties of conformable derivative are given. In section 3, we describe the sinh-Gordon expansion technique. Section 4 is devoted to construct exact solutions of (2+1)-Boiti-Leon Pimponelli equations with time-conformable derivatives. Finally, a conclusion is given in section 5.

2. Conformable derivative

Recently, Khalil and his co-workers [24] presented a novel derivative called conformable. This section is devoted to provide some properties on it.

Definition 1. The conformable derivative with order \(\alpha\) for a function \(f : [0, \infty) \rightarrow R\) is given by

\[
T_\alpha(f)(t) = \lim_{\epsilon \rightarrow 0} \frac{f(t + \epsilon t^{1-\alpha}) - f(t)}{\epsilon}
\]

where \(t > 0, \alpha \in (0, 1)\).

Now, we recall some of its properties:

\[
T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)
\]

for all real constant \(a\) and \(b\),

\[
T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f),
\]

\[
T_\alpha(t^r) = rt^{r-\alpha}
\]

for all \(r\),

\[
T_\alpha \left( \frac{1}{t} \right) = \frac{1}{t^{1+\alpha}},
\]

\[
T_\alpha(C) = 0.
\]

Where \(C\) is a constant.

Moreover, if \(f\) is differentiable, then

\[
T_\alpha(f) = t^{1-\alpha} \frac{df}{dt}(t).
\]

Theorem 1. Suppose that \(f : [0, \infty)\) is differentiable and conformable-differentiable with order \(\alpha\) and the function \(g\) is also differentiable. Then, we have the next property

\[
T_\alpha(fg) = t^{1-\alpha} g(t)f^{(1)}(g(t)).
\]

3. Description of the method

The analytical method, called sinh-Gordon equation expansion technique [25], is an efficient tool
to construct new explicit solutions for many problems arising in various branches of sciences and engineering. The algorithm of this method is based on equation (3) or equation (7) and it can be described as follows

- Consider the following nonlinear equation in the sense of conformable derivative:
  \[ N(u, T^\alpha u, T_x^\alpha u, T_y^\alpha u, ...) = 0. \]  \hspace{1cm} (13)

- Using the following transformation
  \[ u(x, y, t) = U(\xi), \quad \xi = \mu \left( \frac{x}{\alpha} + \frac{y}{\beta} - \frac{\lambda}{\gamma} \right). \]

Equation (13) is converted to an ordinary differential equation

\[ Q(U, U', \mu U', -\lambda U', U'', \mu^2 U'', ..., ) = 0. \] \hspace{1cm} (14)

- Now, we assume that the solution of (14) is as follows
  \[ U(w) = A_0 + \sum_{i=1}^{n} \cosh^{i-1} w [A_i \sinh w + B_i \cosh w], \] \hspace{1cm} (15)

where \( w = w(\xi) \) satisfies (6) or (7) and (13). \( A_i, B_i \) for \( i = 0, 1, 2, ..., n \), are constants to be fixed later.

- By virtue of the balance principle, we take the nonlinear terms and the highest-order derivatives in (14) to determine the value of integer \( n \). Now, let the coefficients of \( \sinh^i w \cosh^j w \) that have same power to be zero, to get a system of equations with the unknowns:

  \( \mu, \lambda, A_i \) and \( B_j \) for \( i = 0, 1, ..., n \).

- Finally, we solve the obtained system with Maple software, then we substitute \( A_0, A_1, B_1, ..., A_n, B_n, \mu \) and \( \lambda \) in (14).

**Remark 1.** When \( m \to 1 \), we have

\[ \text{cs}(\xi, m) \to \text{csch}(\xi), \quad \text{ns}(\xi, m) \to \coth(\xi). \] \hspace{1cm} (16)

Similarly, when \( m \to 0 \), it comes

\[ \text{cs}(\xi, m) \to \cot(\xi), \quad \text{ns}(\xi, m) \to \csc(\xi). \] \hspace{1cm} (17)

4. Application of the method

In this section, we apply the above described method to solve the \((2+1)\)-Boiti-Leon Pimpinelli equations defined as follows (26):

\[ \begin{align*}
T^\alpha_t u_y & = (u^2 - u_x)_{xy} + 2v_{xxx}, \\
T^\alpha_t v_y & = v_{xx} + 2w_x.
\end{align*} \] \hspace{1cm} (18)

Accordingly, we consider the following wave transformation

\[ \begin{align*}
u(x, y, t) & = U(\xi), \\
v(x, y, t) & = V(\xi), \\
\xi & = \mu \left( x + y - \lambda \frac{\alpha}{\gamma} \right),
\end{align*} \] \hspace{1cm} (19)

where \( \mu, \lambda \) are constants to be fixed later.

The previous wave transformation reduces (20) to the following system of ODEs

\[ \begin{align*}
T^\alpha_t (u_y) & = -\lambda \mu^2 U'', \\
(u^2 - u_x)_{xy} & = \mu^2 \left[ (U''')^2 - \mu U''' \right],
\end{align*} \hspace{1cm} (20)

\[ \begin{align*}
2v_{xxx} & = 2\mu^3 V'', \\
T^\alpha_t v & = -\lambda \mu V', \\
v_{xx} & = \mu^2 V'', \\
2w_x & = 2\mu V'.
\end{align*} \]

Then, the new system becomes

\[ \begin{align*}
-\lambda \mu^2 U'' & = \mu^2 (U'')^2 - \mu^3 U''' + 2\mu^3 V'', \\
-\lambda \mu V' & = \mu^2 V'' + 2\mu UV'.
\end{align*} \] \hspace{1cm} (21)

After simplification, we get

\[ -\lambda U'' = (U'')^2 - \mu U''' + 2\mu V'', \] \hspace{1cm} (22)

\[ -\lambda V' = \mu V'' + 2UV'. \] \hspace{1cm} (23)

integrating equation (22) twice and taking zero as constants of integration, yields to

\[ V' = \frac{U'}{2} - \frac{U^2 + \lambda U}{2\mu}. \] \hspace{1cm} (24)

Injecting equation (24) into equation (23), gives the following nonlinear differential equation

\[ \mu^2 U'' - 2U^3 - 3\lambda U^2 - \lambda^2 U = 0. \] \hspace{1cm} (25)
Now, balancing the terms \( U'' \) and \( U^3 \), yields \( n = 1 \). Therefore, the solutions of equation (25) is converted to the following form

\[
U(\xi) = A_0 + A_1 \sinh (w(\xi)) + B_1 \cosh (w(\xi)).
\]

(26)

Substituting (26) into (25), we get a set of algebraic equations for \( \lambda, \mu, A_0, A_1, \) and \( B_1 \) as follows

\[
\begin{align*}
\text{eq1} & = -6A_1^2B_1 - 2B_1^3 + 2B_1 \mu^2, \\
\text{eq2} & = -2A_1^3 - 6A_1B_1^2 + 2A_1 \mu^2, \\
\text{eq3} & = -6A_0 A_1^2 - 6A_0 B_1^2 - 3A_1^2 \lambda - 3B_1^2 \lambda, \\
\text{eq4} & = -12 A_0 A_1 B_1 - 6 A_1 B_1 \lambda, \\
\text{eq5} & = B_1 \mu^2 - 6 A_0^2 B_1 - 6 A_0 B_1 \lambda + 6 A_1^2 B_1 - 2B_1 \mu^2 - B_1 \lambda^2, \\
\text{eq6} & = A_1 \mu^2 - 6 A_0^2 A_1 - 6 A_0 A_1 \lambda + 2A_1^3 - A_1 \mu^2 - A_1 \lambda^2, \\
\text{eq7} & = -2A_0^3 - 3 \lambda A_0^2 + 6 A_0 A_1^2 - \lambda^2 A_0 + 3 \lambda A_1^2.
\end{align*}
\]

(27)

Solving the set of above equations, we get

**Case I:**

\[
\begin{align*}
A_0 & = -\frac{\lambda}{2}, & B_1 & = \frac{\lambda}{\sqrt{2m^2+2}}, \\
\mu & = -\frac{\lambda}{\sqrt{2m^2+2}}, & A_1 & = 0.
\end{align*}
\]

By using (28) and (29), we attain

\[
U_1(\xi) = -\frac{1}{2} \lambda + \frac{\lambda \text{ns}(\xi, m)}{\sqrt{2m^2+2}}
\]

(28)

and

\[
U_2(\xi) = -\frac{1}{2} \lambda + \frac{\lambda \text{cs}(\xi, m)}{\sqrt{2m^2-4}}
\]

(31)

and

\[
V_2(\xi) = -\frac{1}{2} \lambda + \frac{\lambda \text{ns}(\xi, m)}{\sqrt{2m^2-4}} + 1/8 \frac{\lambda^2 \xi}{\mu}
\]

\[+1/4 \frac{\lambda^2 \sqrt{2m^2-4} \ln(\text{ns}(\xi, m) - \text{ds}(\xi, m))}{\mu (m^2-2)}
\]

\[-1/2 \frac{\lambda^2 \ln(\text{ns}(\xi, m) - \text{ds}(\xi, m))}{\mu (m^2-4)},
\]

(32)

where \( \xi = \mu (x + y - \frac{\mu}{\alpha}). \)

**Case III:**

\[
\begin{align*}
A_0 & = -\frac{1}{2} \lambda, & A_1 & = \frac{1}{2} \frac{\lambda}{\sqrt{2m^2-1}}, \\
B_1 & = \frac{1}{2} \frac{\lambda}{\sqrt{2m^2-1}}, & \mu & = \frac{\lambda}{\sqrt{2m^2-1}}.
\end{align*}
\]

(33)

By using (33) and (26), we get

\[
U_3(\xi) = -\frac{1}{2} \lambda + \frac{\lambda \text{cs}(\xi, m)}{\sqrt{2m^2-1}} + \frac{1}{2} \frac{\lambda \text{ns}(\xi, m)}{\sqrt{2m^2-1}}
\]

(34)

where \( \xi = \mu (x + y - \frac{\mu}{\alpha}). \)

**Remark 2.** The expression of \( V_3 \) is too long to be mentioned here.

If \( m \to 0 \), the following solitary wave solutions of (20) are generated from (23), (31) and (34), namely

\[
U_4(\xi) = -\frac{1}{2} \lambda + \frac{1}{2} \lambda \text{cs}(\xi, \sqrt{2})
\]

(35)

\[
V_4(\xi) = \frac{1}{4} \lambda \text{cs}(\xi, \sqrt{2}) - \frac{1}{8} \lambda \sqrt{2}\xi
\]

\[-\frac{1}{2} \lambda \ln(\text{cs}(\xi) - \text{cot}(\xi))
\]

\[-\frac{1}{4} \lambda \sqrt{2} \text{cos}(\xi) - \frac{1}{2} \lambda \ln(\text{cs}(\xi) + \text{cot}(\xi)),
\]

(36)

\[
U_5(\xi) = -\frac{1}{2} \lambda - \frac{1}{2} i \lambda \text{cot}(\xi),
\]

(37)
where $\xi = \mu (x + y - \lambda \frac{w}{m})$.

If $m \to 1$, we get from (28), (31) and (34), new solutions of (20)

$$U_7(\xi) = -\frac{1}{2} \lambda + \frac{1}{2} \lambda \coth(\xi),$$

$$V_7(\xi) = -1/4 \lambda \xi + 1/8 \lambda \ln(\cosh(\xi) - \sinh(\xi)) + 3/8 \lambda \ln(\cosh(\xi) + \sinh(\xi)),$$

$$U_8(\xi) = -\frac{1}{2} \lambda - \frac{1}{2} i \lambda \csc(\xi) \sqrt{2},$$

$$V_8(\xi) = -1/4 i \lambda \sqrt{2} \csch(\xi) - 1/8 i \lambda \sqrt{2} \xi + \lambda \text{arctanh} \left( e^{\xi} \right) + \frac{1/4 i \lambda \sqrt{2} \cosh(\xi)}{\csch(\xi) + 1/2 \lambda \ln \left( \frac{\cosh(\xi) - 1}{\sinh(\xi)} \right)},$$

$$U_9(\xi) = -\frac{1}{2} \lambda + \frac{1}{2} \lambda \csch(\xi) + \frac{1}{2} \lambda \coth(\xi),$$

$$V_9(\xi) = 1/4 \lambda \csch(\xi) + 1/4 \lambda \xi + 3/8 \lambda \coth(\xi) + \lambda/8 \left( \ln(\coth(\xi) - 1) - \ln(\coth(\xi) + 1) \right) - 1/2 \lambda \text{arctanh} \left( e^{\xi} \right) + 1/8 \lambda \frac{\cosh(\xi)}{\sinh(\xi)} + 1/4 \frac{\lambda(\cosh(\xi))^2}{\sinh(\xi)} - 1/4 \lambda \sinh(\xi) - 1/4 \lambda \ln \left( \frac{\cosh(\xi) - 1}{\sinh(\xi)} \right),$$

where $\xi = \mu (x + y - \lambda \frac{w}{m})$.


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